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AKSZ constructions for topological membranes on G_2 -manifolds

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Abstract

We consider AKSZ constructions of BV actions for closed topological membranes, and their dimensional reductions to topological string sigma-models. Two inequivalent AKSZ constructions for topological membranes on G_2 -manifolds are proposed, in each of which the two existing topological membrane theories appear as different gauge fixed versions. Their dimensional reductions give new AKSZ constructions for the topological A-model, which on further dimensional reduction gives an AKSZ formulation of supersymmetric quantum mechanics. We show that the two AKSZ membrane models originate through worldvolume dimensional reduction of a single AKSZ threebrane theory, which gives the standard 2-Courant bracket as the underlying derived bracket. Double dimensional reduction of the twisted topological threebrane theory on a circle yields the standard Courant sigma-model for string theory with NS–NS flux.

Contents

1	Introduction	2
2	Aspects of the AKSZ construction	3
2.1	AKSZ sigma-models	3
2.2	Gauge fixing in the superfield formalism	6
2.3	B -fields and the Poisson sigma-model	8
2.4	Courant algebroids and topological membranes	9
2.5	Lie algebroids up to homotopy and topological threebranes	12
2.6	Dimensional reduction and effective actions	14
2.7	Double dimensional reduction of twisted sigma-models	15
3	AKSZ theories of topological membranes on G_2-manifolds	16
3.1	Topological membrane theories	16
3.2	BV formulation and AKSZ constructions	18
3.3	Derived brackets	20
3.4	Dimensional reductions from topological threebrane theories	21
4	AKSZ theories for the topological A-model	23
4.1	Dimensional reduction of AKSZ membrane sigma-models	23
4.2	The topological A-model	25
4.3	BV formulation and AKSZ constructions	26
4.4	Dimensional reduction from the standard Courant sigma-model	27
5	AKSZ theory for supersymmetric quantum mechanics	28
5.1	Dimensional reduction of the A-model	28
5.2	Supersymmetric quantum mechanics	29
5.3	AKSZ construction	30
6	Conclusions and outlook	31

1 Introduction

Topological M-theory was originally proposed as a unification of the topological A- and B-models [1, 2], and is intended to capture a topological sector of physical M-theory. It can be constructed on seven-dimensional manifolds of G_2 -holonomy where it has reduced $\mathcal{N} = 1$ supersymmetry. The theory of [1] is based on a Hitchin-type form theory of G_2 -manifolds, and its dimensional reduction on a circle gives Hitchin's form theories of the topological A- and B-models.

The A- and B-models have worldsheet formulations as string theories where they are given by two-dimensional topological sigma-models. Hence it is natural to expect that topological M-theory has a worldvolume formulation and its fundamental objects are topological membranes. Two different membrane theories have been proposed for this purpose. One is constructed using the Mathai-Quillen formalism in [3], which reduces on a circle to the Mathai-Quillen construction of the A-model [4, 5] and its path integral localizes on associative three-cycles. The other one is constructed in [6] as a BRST gauge fixed version of the simple topological action constructed by pullback to the membrane worldvolume of the harmonic three-form associated to the G_2 -structure, which also reduces to the A-model and localizes on associative three-cycles.

Both types of topological membranes are intended to be the fundamental objects of the same theory, which inevitably raises the question of whether they can be described within a single membrane model. In this paper we aim to give a unified treatment of these objects by describing some new aspects of the AKSZ formulation for topological membranes and topological string sigma-models. AKSZ formulations provide natural geometric methods for constructing BV quantized sigma-models which circumvent the difficulties involved in finding the BV extension of a classical action with degenerate symmetries [7–10]. They produce examples of topological field theories of Schwarz-type in arbitrary dimensionality which include well-known cases such as Chern-Simons theory, BF-theory and the Poisson sigma-model. Restrictions to special gauge fixing action functionals also yield examples of topological field theories of Witten-type, including the A/B-models and Rozansky-Witten theory. In this paper we propose two different BV quantized sigma-models for topological membranes on G_2 -manifolds given by the AKSZ formulation, which each give back the membrane theories discussed above in particular gauges. Our distinct AKSZ membrane theories have the special feature that they can be unified within a single AKSZ threebrane sigma-model, in which the derived bracket is the same as the anomaly-free current algebra of topological membranes induced on the generalized tangent bundle $T \oplus \bigwedge^2 T^*$ of G_2 -manifolds [11].

One of the main motivations for studying AKSZ constructions is in the context of fluxes and generalized geometry in string theory and M-theory. Fluxes in compactifications of type II string theory appear as twists of the Courant algebroid structure of the T-duality inspired generalized tangent bundle [12–15]. Courant algebroids are in one-to-one correspondence to three-dimensional topological AKSZ sigma-models with target QP-manifold of degree 2, which are called Courant sigma-models [16–22]. Courant sigma-models geometrize fluxes in the sense that they are uplifts of string sigma-models to one higher dimension which can accommodate fluxes [9, 10, 23, 24].

The topological A- and B-models can also be described by gauge fixed AKSZ sigma-models. They have been extensively studied by introducing AKSZ membrane models with generalized complex structures arising from generalized geometry, which reduce to the A- and B-models [26–30]. In forthcoming work [31] we will show that the A-model is closely related to the contravariant Courant sigma-model of [25].

Inspired by this analogy, in this paper we construct AKSZ membrane sigma-models which dimensionally reduce to give new AKSZ constructions for the A-model after gauge fixing and canonical transformation. Performing a further dimensional reduction of one of these string models then gives a novel AKSZ construction for supersymmetric quantum mechanics. We also propose an AKSZ

topological three-brane theory, which reproduces our membrane sigma-models through a worldvolume dimensional reduction, and yields the standard 2-Courant bracket as its derived bracket, which fits it into the context of exceptional generalized geometry in M-theory. We also study the three-brane theory with a four-form flux twisting, and show that upon double dimensional reduction on a circle it reproduces the standard H -flux twisted Courant sigma-model.

This paper is organized as follows. In §2 we give a relatively detailed overview of various salient features of the AKSZ construction, together with a few examples of relevance for this paper such as the Poisson, Courant and 2-Courant sigma-models, and we derive the correspondence between flux twistings of these topological field theories. In §3 we introduce two AKSZ constructions for each of the topological membrane theories of [3] and [6], and describe their underlying derived bracket algebra along with their origins as dimensional reductions of AKSZ topological threebrane theories. In §4 we calculate their dimensional reductions and show that the reduced AKSZ sigma-models are AKSZ constructions for the topological A-model, which can be similarly unified through worldvolume dimensional reduction of a single Courant sigma-model. We dimensionally reduce it further in §5 and get an AKSZ construction for supersymmetric quantum mechanics. Finally, we close with some concluding remarks and outlook on further applications of our constructions in §6.

2 Aspects of the AKSZ construction

In this section we survey some pertinent background about the AKSZ construction and BV quantization, and describe several relevant examples that we will encounter throughout this paper.

2.1 AKSZ sigma-models

We begin by briefly introducing the ingredients of AKSZ theory. A more complete review can be found in [10]. The AKSZ construction is a BV quantized sigma-model formulation, and it gives a geometric solution to the classical master equation

$$(\mathcal{S}, \mathcal{S})_{\text{BV}} = 0 \tag{2.1}$$

given by the BV bracket, which imposes BRST symmetry. The solution \mathcal{S} is called the AKSZ action, which is just a BV action.

Two classes of supermanifolds enter the AKSZ formalism. The ‘source’ $(\mathcal{W}, Q_{\mathcal{W}}, \mu)$ consists of a differential graded (dg-)manifold, i.e. a graded manifold \mathcal{W} equipped with a cohomological vector field¹ $Q_{\mathcal{W}}$, and a measure μ which is invariant under $Q_{\mathcal{W}}$. The ‘target’ $(\mathcal{M}, Q_{\gamma}, \omega)$ is a symplectic dg-manifold, i.e. a graded manifold \mathcal{M} with a cohomological vector field Q_{γ} , and a graded symplectic form ω for which Q_{γ} is a Hamiltonian vector field:² $\iota_{Q_{\gamma}}\omega = d\gamma$ for some Hamiltonian function γ on \mathcal{M} .

In this paper we are interested in the construction of d -dimensional topological sigma-models for closed branes. Hence we take $\mathcal{W} = T[1]\Sigma_d$, the tangent bundle of a d -dimensional closed and oriented worldvolume manifold Σ_d with the degree of its fibers shifted by 1, which is isomorphic to the exterior algebra of differential forms $(\Omega(\Sigma_d), \wedge)$. We choose the cohomological vector field $Q_{\mathcal{W}}$ corresponding to the de Rham differential, which in local affine coordinates $\hat{z}^{\hat{\mu}} = (\sigma^{\mu}, \theta^{\mu}) \in T[1]\Sigma_d$, with degree 0 coordinates σ^{μ} on Σ_d and degree 1 fiber coordinates θ^{μ} , has the form³ $Q_{\mathcal{W}} = \theta^{\mu} \frac{\partial}{\partial \sigma^{\mu}} =: D$. The measure in local coordinates can be written in the form $\mu = d^d \hat{z} := d^d \sigma d^d \theta$.

¹A vector field Q is cohomological if it is of degree 1 and its Lie derivative \mathcal{L}_Q squares to zero.

²The operation ι_Q denotes contraction of a differential form along the vector field Q .

³Repeated upper and lower indices are always implicitly understood to be summed over.

The AKSZ space of fields is a mapping space

$$\mathcal{M} = \text{Map}(T[1]\Sigma_d, \mathcal{M}) \quad (2.2)$$

consisting of smooth maps from $(\mathcal{W}, Q_{\mathcal{W}}, \mu)$ to $(\mathcal{M}, Q_{\gamma}, \omega)$. In order to reproduce the BV formalism, the symplectic structure ω is taken to be of degree $d-1$, so that the Hamiltonian function γ is of degree d . We can introduce local coordinates on \mathcal{M} via the superfields

$$\hat{X}^i(\hat{z}^{\hat{\mu}}) = \phi^*(\hat{X}^i)(\hat{z}^{\hat{\mu}}), \quad (2.3)$$

for local coordinates $\hat{z}^{\hat{\mu}} \in \mathcal{W}$, $\hat{X}^i \in \mathcal{M}$ and $\phi \in \mathcal{M}$. Then the de Rham differential on \mathcal{M} is given for $\mathcal{W} = T[1]\Sigma_d$ by the vector field

$$\delta = (-1)^d \int_{T[1]\Sigma_d} d^d \hat{z} \, \delta \hat{X}^i(\hat{z}) \frac{\vec{\delta}}{\delta \hat{X}^i(\hat{z})} \quad (2.4)$$

with ghost number 1. The cohomological vector fields $Q_{\mathcal{W}}$ and Q_{γ} induce a cohomological vector field Q on \mathcal{M} in the following way. For $\phi \in \mathcal{M}$ and $\hat{z} \in \mathcal{W}$, define

$$(Q_0 \phi)(\hat{z}) = d\phi(\hat{z}) Q_{\mathcal{W}}(\hat{z}) \quad \text{and} \quad (Q_{\gamma} \phi)(\hat{z}) = Q_{\gamma}(\phi(\hat{z})). \quad (2.5)$$

Then \mathcal{M} is a dg-manifold with cohomological vector field

$$Q = Q_0 + Q_{\gamma}. \quad (2.6)$$

Given an n -form $\alpha \in \Omega^n(\mathcal{M})$, we can lift it to an n -form $\alpha \in \Omega^n(\mathcal{M})$ by transgression to the mapping space as

$$\alpha = \int_{\mathcal{W}} \mu \, \text{ev}^*(\alpha), \quad (2.7)$$

where $\text{ev} : \mathcal{W} \times \mathcal{M} \rightarrow \mathcal{M}$ is the evaluation map. This definition allows us to think of α as an n -form functional of the fields in \mathcal{M} , on which the de Rham differential acts as

$$\delta \alpha = \delta \int_{\mathcal{W}} \mu \, \text{ev}^*(\alpha) = \int_{\mathcal{W}} \mu \, \text{ev}^*(d\alpha). \quad (2.8)$$

Due to the integration, α has ghost number $U(\alpha) - d$, where $U(\alpha)$ denotes the internal degree of α . In particular, since transgression is a chain map, from the symplectic form ω on \mathcal{M} and a Liouville potential ϑ , such that $\omega = d\vartheta$, we get the symplectic form ω of degree -1 and Liouville potential ϑ on \mathcal{M} , such that $\omega = \delta \vartheta$. Furthermore, the cohomological vector field Q on \mathcal{M} is also Hamiltonian with Hamiltonian function $-\iota_{Q_0} \vartheta + \gamma$ of degree 0: $\iota_{Q_{\gamma}} \omega = \delta \gamma$. In other words, the mapping space of superfields \mathcal{M} is itself a symplectic dg-manifold.

The BV bracket $(\cdot, \cdot)_{\text{BV}}$ is the graded Poisson bracket of degree 1 on \mathcal{M} defined from ω , and it corresponds to the graded Poisson bracket $\{\cdot, \cdot\}$ of degree $-d+1$ on \mathcal{M} defined from ω , since the transgression map $\int_{\mathcal{W}} \mu \, \text{ev}^*$ is a Lie algebra homomorphism from $(\mathcal{M}, \{\cdot, \cdot\})$ to $(\mathcal{M}, (\cdot, \cdot)_{\text{BV}})$:

$$\int_{\mathcal{W}} \mu \, \text{ev}^*(\{F, G\}) = \left(\int_{\mathcal{W}} \mu \, \text{ev}^*(F), \int_{\mathcal{W}} \mu \, \text{ev}^*(G) \right)_{\text{BV}}, \quad (2.9)$$

where F and G are any local functions on \mathcal{M} . In particular, the cohomological vector fields can be represented through derived brackets as

$$Q_{\gamma} = \{\gamma, \cdot\} \quad \text{and} \quad Q = D + (\gamma, \cdot)_{\text{BV}}, \quad (2.10)$$

and the cocycle conditions $Q_\gamma^2 = 0$ and $\mathbf{Q}^2 = 0$ are equivalent to $\{\gamma, \gamma\} = 0$ and $(\gamma, \gamma)_{\text{BV}} = 0$.

It is now evident how to construct the desired BV action. For this, we choose a Liouville potential ϑ on \mathcal{M} , and write the solution of the classical master equation, i.e. the degree 0 AKSZ action \mathbf{S} , on \mathcal{M} in the form

$$\mathbf{S} = \mathbf{S}_0 + \gamma , \quad (2.11)$$

where $\mathbf{S}_0 = -\iota_{Q_0}\vartheta$ is the kinetic term and the Hamiltonian function γ on \mathcal{M} is the interaction term. For a source superworldvolume $\mathcal{W} = T[1]\Sigma_d$, one has explicitly

$$\mathbf{S} = \int_{T[1]\Sigma_d} d^d \hat{z} \left(-\iota_{Q_{\mathcal{W}}} \text{ev}^*(\vartheta) + \text{ev}^*(\gamma) \right) . \quad (2.12)$$

In the BV formalism, the cohomological vector field \mathbf{Q} corresponds to the BRST charge which generates BRST transformations of superfields and the BRST cohomology of the mapping space \mathcal{M} .

Canonical transformations. In the BV formalism, the phase space \mathcal{M} of superfields is only defined modulo canonical transformations. These are the maps of \mathcal{M} that leave the BV bracket structure invariant. A canonical transformation is associated to a degree $d-1$ function α on \mathcal{M} . We use the notation $\delta_\alpha := \{\cdot, \alpha\}$ for the corresponding Hamiltonian vector field, and e^{δ_α} and e^{δ_α} for the respective canonical transformations given by the adjoint actions of the respective Poisson brackets. The action of the canonical transformation on γ is given by $e^{\delta_\alpha}\gamma = \int_{\mathcal{W}} \mu \text{ev}^*(e^{\delta_\alpha}\gamma)$, with $e^{\delta_\alpha}\gamma = \gamma + Q_\gamma\alpha + O(\alpha^2)$, which preserves the classical master equation as

$$\{e^{\delta_\alpha}\gamma, e^{\delta_\alpha}\gamma\} = e^{\delta_\alpha}\{\gamma, \gamma\} = 0 , \quad (2.13)$$

due to $\{\gamma, \gamma\} = 0$. Then the AKSZ action $\mathbf{S}_0 + \gamma$ is equivalent to $\mathbf{S}_0 + e^{\delta_\alpha}\gamma$ up to a canonical transformation, which at first order shifts the action by a BRST-exact term.

A duality transformation in the AKSZ formalism is defined as a symplectomorphism \mathbf{f} , which is a diffeomorphism between underlying symplectic manifolds

$$\mathbf{f} : (\mathcal{M}, \omega) \longrightarrow (\mathcal{M}', \omega') , \quad (2.14)$$

satisfying

$$\mathbf{f}^*\omega' = \omega . \quad (2.15)$$

In other words, \mathbf{f} is a coordinate transformation on symplectic manifolds which leaves the symplectic structure invariant. Then the canonical transformation e^{δ_α} is a duality transformation as well.

QP-manifolds. A common choice of target for the AKSZ construction is to take \mathcal{M} to be an N-manifold, which is a graded manifold with no coordinates of negative degree. In this case the triple $(\mathcal{M}, Q_\gamma, \omega)$ is called a QP-manifold of degree $n = d-1$; if the N-manifold \mathcal{M} is concentrated in degrees $0, 1, \dots, n$, then $(\mathcal{M}, Q_\gamma, \omega)$ is called a symplectic Lie n -algebroid, and it arises from an n -graded vector bundle over the degree 0 body $M = \mathcal{M}_0$ of \mathcal{M} [10]; in particular, functions of degree $n-1$ can be identified with sections of a vector bundle $E \rightarrow M$ equipped with the structure of a Leibniz algebroid. For example, in the simplest dimension $d = 1$ with target a degree 0 QP-manifold, one necessarily has $Q_\gamma = 0$ and thus a symplectic Lie 0-algebroid is just an ordinary symplectic manifold (\mathcal{M}, ω) ; in this case the degree 1 Hamiltonian function γ is locally constant on \mathcal{M} and the AKSZ construction produces a topological quantum mechanics given as a one-dimensional Chern-Simons theory whose Chern-Simons form is a Liouville potential ϑ [32, 33].

In the following we will describe the AKSZ topological field theories associated with the first few non-trivial members in the hierarchy of QP-structures on the target manifold for dimensions $d = 2, 3, 4$, in the context of the string and membrane models of interest in this paper. Later on we shall also deal with targets that have negative degree coordinates and hence unravel new constructions even in low dimension.

2.2 Gauge fixing in the superfield formalism

The entire field content of a system with degenerate symmetries is usually specified by separating it into ‘fields’, which includes the original physical and ghost fields from the BRST picture, and dual ‘antifields’, which correspond to the equations of motion and define canonically conjugate variables with respect to the symplectic phase space structure on the space \mathcal{M} of all fields. In AKSZ constructions the fields and antifields are not distinguished from the onset. The theory is specified once the antifields are assigned, and different choices yield different field theories.

In the usual BV quantized theories, the fields and antifields are distinguished from the start. One chooses a gauge fixing fermion $\Psi[\phi]$, which is a functional of the fields ϕ^a (but not the antifields) of ghost number $U = -1$, and then the antifields ϕ_a^+ are fixed to the variations $\phi_a^+ = \frac{\delta \Psi}{\delta \phi^a}$. This can be reformulated in terms of the BV symplectic structure on the space of superfields \mathcal{M} . For this, we consider the case where the source dg-manifold is the superworldvolume $\mathcal{W} = T[1]\Sigma_d$ with local coordinates $\hat{z} = (\sigma, \theta)$ and write a generic BV symplectic structure on superfields in its canonical form as

$$\omega = \int_{T[1]\Sigma_d} d^d \hat{z} \, \delta \phi_a^+(\hat{z}) \delta \phi^a(\hat{z}) , \quad (2.16)$$

where we chose a convenient ordering of antifields ϕ_a^+ and fields ϕ^a in this way. We write $|a|$ for the degree of the superfield ϕ^a ; then its antifield ϕ_a^+ has degree $d - 1 - |a|$. If the Liouville potential is chosen as

$$\vartheta = \int_{T[1]\Sigma_d} d^d \hat{z} \, \phi_a^+(\hat{z}) \delta \phi^a(\hat{z}) , \quad (2.17)$$

then the kinetic part of the AKSZ action is

$$S_0 = \int_{T[1]\Sigma_d} d^d \hat{z} \, (-1)^{|a|} \phi_a^+(\hat{z}) D \phi^a(\hat{z}) . \quad (2.18)$$

We choose a gauge fixing fermion $\Psi[\phi]$, which is a functional on superfields $\phi(\hat{z}) \in \mathcal{M}$, and fix the antifields to

$$\phi_a^+(\hat{z}) = (-1)^{|a|(d+1)} \frac{\vec{\delta} \Psi}{\delta \phi^a(\hat{z})} , \quad (2.19)$$

where an extra sign factor has been introduced, which depends on the dimension of the worldvolume. The left-acting functional derivative is defined in the usual way by

$$\lim_{\epsilon \rightarrow 0} \frac{\Psi[\phi + \epsilon \xi] - \Psi[\phi]}{\epsilon} =: \int_{T[1]\Sigma_d} d^d \hat{z} \, \xi(\hat{z}) \frac{\vec{\delta} \Psi}{\delta \phi(\hat{z})} . \quad (2.20)$$

The BV symplectic form (2.16) in the gauge that is fixed by $\Psi[\phi]$ according to (2.19) is

$$\begin{aligned} \omega_\Psi &= \sum_a (-1)^{|a|(d+1)} \int_{T[1]\Sigma_d} d^d \hat{z}_1 \, \delta \frac{\vec{\delta} \Psi}{\delta \phi^a(\hat{z}_1)} \delta \phi^a(\hat{z}_1) \\ &= (-1)^{d+1} \sum_{a,b} (-1)^{|b|(|a|+1)+|a|d} \int_{T[1]\Sigma_d} d^d \hat{z}_1 \int_{T[1]\Sigma_d} d^d \hat{z}_2 \, \frac{\vec{\delta}^2 \Psi}{\delta \phi^b(\hat{z}_2) \delta \phi^a(\hat{z}_1)} \delta \phi^b(\hat{z}_2) \delta \phi^a(\hat{z}_1) . \end{aligned} \quad (2.21)$$

Interchanging variables and indices yields sign changes which are given by

$$\begin{aligned} \frac{\vec{\delta}^2 \Psi}{\delta \phi^b(\hat{z}_2) \delta \phi^a(\hat{z}_1)} &= (-1)^{(|a|+d)(|b|+d)} \frac{\vec{\delta}^2 \Psi}{\delta \phi^a(\hat{z}_1) \delta \phi^b(\hat{z}_2)} , \\ \delta \phi^b(\hat{z}_2) \delta \phi^a(\hat{z}_1) &= (-1)^{(|a|+1)(|b|+1)} \delta \phi^a(\hat{z}_1) \delta \phi^b(\hat{z}_2) , \\ \int_{T[1]\Sigma_d} d^d \hat{z}_1 \int_{T[1]\Sigma_d} d^d \hat{z}_2 &= (-1)^d \int_{T[1]\Sigma_d} d^d \hat{z}_2 \int_{T[1]\Sigma_d} d^d \hat{z}_1 . \end{aligned} \quad (2.22)$$

This shows that the gauge fixed BV symplectic form is a product of a symmetric and an antisymmetric expression, and hence $\omega_\Psi = 0$. Thus gauge fixing with a fermion in the sense of (2.19) means that one chooses a Lagrangian submanifold \mathcal{L} of the space of all fields \mathcal{M} , i.e. a subspace $\mathcal{L} \subset \mathcal{M}$ on which the symplectic form ω vanishes and which has half the dimension of \mathcal{M} . In the following we use this prescription generally: A choice of gauge in BV quantization is equivalent to a choice of a Lagrangian submanifold \mathcal{L} in \mathcal{M} . The Batalin-Vilkovisky theorem [34] ensures that the path integral over \mathcal{L} is independent of the choice of representative for the homology class of the Lagrangian submanifold \mathcal{L} . By the localization theorem, the path integral localizes over the fixed point locus of the BV-BRST charge Q in the Lagrangian subspace \mathcal{L} . From a physical point of view, the Lagrangian submanifold intersects the gauge orbits orthogonally, i.e. the action of the BV-BRST charge $(S, \cdot)_{\text{BV}}$ vanishes on Lagrangian submanifolds, as the BV bracket acts as zero there. Thus the BV gauge symmetry is completely fixed on Lagrangian submanifolds.⁴

Let us now reformulate these observations in terms of the expansion coefficients of superfields. An arbitrary superfield ϕ^a can be expanded in terms of the degree 1 fiber coordinates θ^μ of $\mathcal{W} = T[1]\Sigma_d$ in the form

$$\phi^a(\hat{z}) = \phi^{(0)a}(\sigma) + \phi_{\mu_1}^{(1)a}(\sigma) \theta^{\mu_1} + \frac{1}{2} \phi_{\mu_1 \mu_2}^{(2)a}(\sigma) \theta^{\mu_1} \theta^{\mu_2} + \dots + \frac{1}{d!} \phi_{\mu_1 \dots \mu_d}^{(d)a}(\sigma) \theta^{\mu_1} \dots \theta^{\mu_d} , \quad (2.23)$$

where $\phi^{(p)a}$ are the degree $|a| - p$ coefficients of ϕ^a which can be identified with p -forms on Σ_d . The BV symplectic form can be written as an integral over the original worldvolume Σ_d as

$$\begin{aligned} \omega &= \sum_{p=0}^d \int_{\Sigma_d} \delta \phi_a^{(p)+} \wedge \delta \phi^{(p)a} \\ &= \sum_{p=0}^d \frac{1}{p!} \int_{\Sigma_d} d^d \sigma \sum_a (-1)^{|a|+p} \delta \tilde{\phi}_a^{(p)+; \mu_1 \dots \mu_p}(\sigma) \delta \phi_{\mu_1 \dots \mu_p}^{(p)a}(\sigma) , \end{aligned} \quad (2.24)$$

where $\tilde{\phi}_a^{(p)+}$ is the dual antifield of $\phi^{(p)a}$ defined by

$$\tilde{\phi}_a^{(p)+; \mu_{d-p+1} \dots \mu_d} = (-1)^{d(d+1+|a|+p)+|a|(p+1)+p} \frac{1}{(d-p)!} \epsilon^{\mu_1 \dots \mu_d} (\phi_a^+)^{(d-p)}_{\mu_1 \dots \mu_{d-p}} . \quad (2.25)$$

Here $\epsilon^{\mu_1 \dots \mu_d}$ is the Levi-Civita tensor density on Σ_d , and $(\phi_a^+)^{(d-p)}$ are the expansion coefficients of the superfield ϕ_a^+ . The BV symplectic form with this sign convention gives the canonical Poisson bracket relations

$$\{ \phi_{\mu_1 \dots \mu_p}^{(p)a} , \tilde{\phi}_b^{(p') +; \nu_1 \dots \nu_{p'}} \} = \delta^{pp'} \delta^a_b \delta_{\mu_1 \dots \mu_p}^{\nu_1 \dots \nu_{p'}} . \quad (2.26)$$

Gauge fixing with a fermion $\Psi[\phi]$ then takes the more familiar form

$$\tilde{\phi}_a^{(p)+} = \frac{\vec{\delta} \Psi}{\delta \phi^{(p)a}} , \quad (2.27)$$

and it gives a vanishing symplectic structure $\omega_\Psi = 0$, whence the fermion Ψ generates a Lagrangian submanifold in the terms of the expansion coefficients as well.

⁴We have not studied the Gribov problem in this context.

2.3 B -fields and the Poisson sigma-model

In dimension $d = 2$, the AKSZ theory with target space a degree 1 QP-manifold describes the topological sigma-model for closed strings in an NS-NS B -field background. In the worldsheet sigma-model approach, the fundamental field is a map $X : \Sigma_2 \rightarrow M$ from a closed and oriented Riemann surface Σ_2 to a target space M . Denoting the local coordinates by $(X^i) \in M$ and $(\sigma^\mu) \in \Sigma_2$, the string field X is described by a set of functions $(X^i(\sigma^\mu))$ on Σ_2 . The topological part of the bosonic string action is

$$I_{\Sigma_2, B} = \int_{\Sigma_2} X^*(B) = \frac{1}{2} \int_{\Sigma_2} B_{ij} dX^i \wedge dX^j, \quad (2.28)$$

where $B = \frac{1}{2} B_{ij} dX^i \wedge dX^j$ is the Kalb-Ramond two-form field on M . If B is non-degenerate, it corresponds to an almost symplectic structure on M and we can write the classically equivalent first order string sigma-model

$$I_{\Sigma_2, \pi}^{(1)} = \int_{\Sigma_2} \left(\chi_i \wedge dX^i + \frac{1}{2} \pi^{ij} \chi_i \wedge \chi_j \right), \quad (2.29)$$

where B_{ij} is the inverse of $-\pi^{ij}$ and $\chi = (\chi_i) \in \Omega^1(\Sigma_2, X^*T^*M)$ is an auxiliary one-form. The bivector $\pi = \frac{1}{2} \pi^{ij} \frac{\partial}{\partial X^i} \wedge \frac{\partial}{\partial X^j}$ on M is a Poisson bivector on-shell, which is equivalent to a flat B -field $dB = 0$, so that the Kalb-Ramond field corresponds to a symplectic structure on M . This is the action functional of the Poisson sigma-model [35, 36].

The AKSZ formulation of the Poisson sigma-model is studied in [8]. We take $\mathcal{W} = T[1]\Sigma_2$, and $\mathcal{M} = T^*[1]M$ with⁵ degree 0 base coordinates X^i on M and degree 1 fiber coordinates χ_i . The canonical symplectic form on \mathcal{M} is

$$\omega = d\chi_i \wedge dX^i, \quad (2.30)$$

which leads to the canonical graded Poisson bracket $\{\chi_i, X^j\} = \delta_i^j$ on the local coordinates of \mathcal{M} . We choose the Liouville potential to be $\vartheta = \chi_i dX^i$. The most general form of a degree 2 Hamiltonian function γ on \mathcal{M} is given by a (possibly degenerate) $(0, 2)$ -tensor $\pi = \pi^{ij} \frac{\partial}{\partial X^i} \otimes \frac{\partial}{\partial X^j}$ on M as

$$\gamma = \frac{1}{2} \pi^{ij}(X) \chi_i \chi_j. \quad (2.31)$$

The corresponding cohomological vector field Q_γ on \mathcal{M} is

$$Q_\gamma = \pi^{ij} \chi_j \frac{\partial}{\partial X^i} + \frac{1}{2} \frac{\partial \pi^{ij}}{\partial X^k} \chi_i \chi_j \frac{\vec{\partial}}{\partial \chi_k}. \quad (2.32)$$

Compatibility of Q_γ with ω implies $\pi \in \Gamma(\wedge^2 TM)$ and the classical master equation $\{\gamma, \gamma\} = 0$ implies that $\pi = \frac{1}{2} \pi^{ij} \frac{\partial}{\partial X^i} \wedge \frac{\partial}{\partial X^j}$ must be a Poisson bivector on M , i.e. $\pi^{[i} \frac{\partial \pi^{j]k}}{\partial X^l} = 0$. In other words, a symplectic Lie 1-algebroid is the same thing as a Poisson manifold (M, π) , which by construction is also a Lie algebroid on the cotangent bundle T^*M . The Hamiltonian function determines a derived bracket which defines a Poisson bracket on $C^\infty(M)$ through

$$\{f, g\}_\pi = \pi^{ij} \frac{\partial f}{\partial X^i} \frac{\partial g}{\partial X^j} = -\{\{f, \gamma\}, g\}. \quad (2.33)$$

⁵In the present context we use the terminology ‘degree’ to mean ghost number. In general the degree means total degree, which is the sum of the ghost number and the form degree on the dg-manifold, but here we have only functions.

The kinetic part \mathcal{S}_0 of the AKSZ action is inherited from the cohomological vector field $Q_{\mathcal{W}}$ on $\mathcal{W} = T[1]\Sigma_2$, and is given by

$$\mathcal{S}_0 = \int_{T[1]\Sigma_2} d^2\hat{z} \, \chi_i \mathbf{D}X^i, \quad (2.34)$$

where as before the superworldsheet differential is $\mathbf{D} = \theta^\mu \frac{\partial}{\partial \sigma^\mu} = Q_{\mathcal{W}}$. The BV bracket has the form

$$(\cdot, \cdot)_{\text{BV}} = \int_{T[1]\Sigma_2} d^2\hat{z} \, \frac{\delta}{\delta \mathbf{X}^i} \wedge \frac{\delta}{\delta \chi_i}, \quad (2.35)$$

where

$$\frac{\delta}{\delta \mathbf{X}^i} \wedge \frac{\delta}{\delta \chi_i} := \frac{\overleftarrow{\delta}}{\delta \mathbf{X}^i} \frac{\overrightarrow{\delta}}{\delta \chi_i} - \frac{\overleftarrow{\delta}}{\delta \chi_i} \frac{\overrightarrow{\delta}}{\delta \mathbf{X}^i}. \quad (2.36)$$

Together these ingredients give the AKSZ action for the Poisson sigma-model as

$$\mathcal{S} = \int_{T[1]\Sigma_2} d^2\hat{z} \left(\chi_i \mathbf{D}X^i + \frac{1}{2} \pi^{ij} \chi_i \chi_j \right), \quad (2.37)$$

where $\mathbf{f} = \phi^*(f) = f(\phi)$ for a function f on \mathcal{M} and $\phi \in \mathcal{M}$. Integrating over the odd coordinates θ^μ and restricting to the degree 0 fields in (2.37) recovers the classical action (2.29), and in this sense the action (2.37) provides a BV quantization of the original Poisson sigma-model. By the fixed point theorem, the path integral localizes onto critical points of the action \mathcal{S} , which are the fixed points of the cohomological vector field Q_γ that defines the Poisson cohomology $H_\pi^\bullet(M)$ of M . This formulation of the Poisson sigma-model also gives an AKSZ construction of the A-model [7, 10, 17, 26, 28, 30].

2.4 Courant algebroids and topological membranes

The next dimension $d = 3$ is particularly relevant to extending the Poisson sigma-model to closed string backgrounds with non-zero NS–NS three-form flux $H = dB$, or to M-theory backgrounds with three-form C -field. In this setting the closed strings are replaced with membranes described by maps $X = (X^i)$ from a closed three-dimensional worldvolume Σ_3 to the target space M . The topological part of the bosonic membrane action is the Wess-Zumino coupling

$$I_{\Sigma_3, H} = \int_{\Sigma_3} X^*(H) = \frac{1}{3!} \int_{\Sigma_3} H_{ijk} dX^i \wedge dX^j \wedge dX^k. \quad (2.38)$$

This action is classically equivalent to the first order membrane sigma-model action

$$I_{\Sigma_3, H}^{(1)} = \int_{\Sigma_3} \left(F_i \wedge (dX^i - \psi^i) - \chi_i \wedge d\psi^i + \frac{1}{3!} H_{ijk} \psi^i \wedge \psi^j \wedge \psi^k \right), \quad (2.39)$$

where $\psi = (\psi^i) \in \Omega^1(\Sigma_3, X^*TM)$ and $\chi = (\chi_i) \in \Omega^1(\Sigma_3, X^*T^*M)$ are one-forms, while $F = (F_i) \in \Omega^2(\Sigma_3, X^*T^*M)$ is an auxiliary two-form. The corresponding AKSZ sigma-model is defined on worldvolume superfields with target space a QP-manifold of degree 2, which corresponds to a Courant algebroid [19].

Recall that a Courant algebroid on a manifold M is a vector bundle E over M equipped with a symmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on its fibers, an anchor map $\rho : E \rightarrow TM$, and a binary bracket of sections $[\cdot, \cdot]_D$, called the Dorfman bracket, which together satisfy

$$\begin{aligned} [e_1, [e_2, e_3]_D]_D &= [[e_1, e_2]_D, e_3]_D + [e_2, [e_1, e_3]_D]_D, \\ \rho(e_1) \langle e_2, e_3 \rangle &= \langle [e_1, e_2]_D, e_3 \rangle + \langle e_2, [e_1, e_3]_D \rangle, \\ \rho(e_1) \langle e_2, e_3 \rangle &= \langle e_1, [e_2, e_3]_D + [e_3, e_2]_D \rangle, \end{aligned} \quad (2.40)$$

where e_1, e_2, e_3 are sections of E . The most common example is the standard Courant algebroid, which features in generalized geometry [37, 38]. It is an extension of the Lie algebroid of tangent vectors by cotangent vectors and is based on the generalized tangent bundle

$$E = TM \oplus T^*M , \quad (2.41)$$

with the three operations

$$\begin{aligned} \langle A + \alpha, B + \beta \rangle &= \iota_A \beta + \iota_B \alpha , \\ \rho(A + \alpha) &= A , \\ [A + \alpha, B + \beta]_D &= [A, B] + \mathcal{L}_A \beta - \iota_B d\alpha , \end{aligned} \quad (2.42)$$

where the sections of $E = TM \oplus T^*M$ are composed of vector fields A, B and one-forms α, β . The NS–NS H -flux then appears as a twisting of the standard Courant algebroid, which gives rise to a deformation of the Dorfman bracket through an extra term as

$$[A + \alpha, B + \beta]_{D,H} = [A, B] + \mathcal{L}_A \beta - \iota_B d\alpha + \iota_A \iota_B H . \quad (2.43)$$

Let us now review the correspondence between Courant algebroids and the AKSZ construction, following [10, 15] for the most part. Given a QP-manifold \mathcal{M} of degree 2, we choose local Darboux coordinates (X^i, ζ^a, F_i) with degrees $(0, 1, 2)$ in which the graded symplectic structure is given as

$$\omega = dF_i \wedge dX^i + \frac{1}{2} k_{ab} d\zeta^a \wedge d\zeta^b . \quad (2.44)$$

Here we have introduced a constant metric k_{ab} on the degree 1 subspace, which is a local coordinate expression of the symmetric pairing in the corresponding Courant algebroid. The graded Poisson brackets of the coordinates are canonical in the sense that

$$\{X^i, F_j\} = \delta^i_j \quad \text{and} \quad \{\zeta^a, \zeta^b\} = k^{ab} , \quad (2.45)$$

where k^{ab} is the inverse of k_{ab} . The most general form of the degree 3 Hamiltonian function γ is given by

$$\gamma = \rho^i_a(X) F_i \zeta^a + \frac{1}{3!} T_{abc}(X) \zeta^a \zeta^b \zeta^c , \quad (2.46)$$

where the functions ρ^i_a and T_{abc} on M give the local forms of the anchor map and fluxes, respectively. The three operations on the Courant algebroid are given by taking derived brackets defined by γ and the graded Poisson bracket through

$$[e_1, e_2]_D = \{\{e_1, \gamma\}, e_2\} , \quad \langle e_1, e_2 \rangle = \{e_1, e_2\} \quad \text{and} \quad \rho(e) = \{e, \{\gamma, \cdot\}\} . \quad (2.47)$$

These operations are defined on degree 1 functions e with local expression $e = f_a(X) \zeta^a$, where f_a is a degree 0 function on the body $M = \mathcal{M}_0$ of \mathcal{M} , which are identified as local sections of a vector bundle E over M . They satisfy the Courant algebroid axioms in (2.40) as a consequence of the classical master equation $\{\gamma, \gamma\} = 0$.

Conversely, given a Courant algebroid on a vector bundle E over M , we define the target QP-manifold \mathcal{M} of degree 2 to be the symplectic submanifold of $T^*[2]E[1]$ corresponding to the isometric embedding $E \hookrightarrow E \oplus E^*$ with respect to the Courant algebroid pairing and the canonical dual pairing. Then X^i are local coordinates on M , F_i are local fiber coordinates of the shifted cotangent bundle $T^*[2]M$, and ζ^a are local fiber coordinates of the shifted vector bundle $E[1]$. In other words, a symplectic Lie 2-algebroid is the same thing as a Courant algebroid.

In this paper we shall only deal with the standard Courant algebroid on the generalized tangent bundle $E = TM \oplus T^*M$. The corresponding QP-manifold of degree 2 is then simply $\mathcal{M} = T^*[2]T[1]M$. The local degree 1 coordinates are dual pairs $\zeta^a = (\psi^i, \chi_i)$ and the symplectic form is

$$\omega = dF_i \wedge dX^i + d\chi_i \wedge d\psi^i. \quad (2.48)$$

For the Liouville potential we choose $\vartheta = F_i dX^i - \chi_i d\psi^i$. The simplest Hamiltonian function with $\rho^i_a = \delta^i_a$ and $T_{abc} = 0$ is given by

$$\gamma_0 = F_i \psi^i, \quad (2.49)$$

and the corresponding cohomological vector field

$$Q_{\gamma_0} = \psi^i \frac{\partial}{\partial X^i} \quad (2.50)$$

corresponds to the de Rham differential on M . Its derived brackets on degree 1 functions

$$A^i \chi_i + \alpha_i \psi^i \longleftrightarrow A^i \frac{\partial}{\partial X^i} + \alpha_i dX^i \quad (2.51)$$

yields the standard Courant bracket which is the antisymmetrization of the standard Dorfman bracket in (2.42) given by

$$[A + \alpha, B + \beta]_C = [A, B] + \mathcal{L}_A \beta - \mathcal{L}_B \alpha - \frac{1}{2} d(\iota_A \beta - \iota_B \alpha). \quad (2.52)$$

The Courant bracket is the natural bracket in generalized geometry which is compatible with the commutator algebra of generalized Lie derivatives [37, 38]. Only the simplest case of pure NS–NS flux $T_{abc} = H_{ijk}$ is consistent with the choice of anchor map $\rho^i_a = \delta^i_a$ of the standard Courant algebroid, which is necessarily closed by the classical master equation. Given a Kalb–Ramond two-form field B on M , with $H = dB$, canonical transformation of the Hamiltonian function (2.49) by the degree 2 function $B = \frac{1}{2} B_{ij}(X) \psi^i \psi^j$ on \mathcal{M} yields the twisted Hamiltonian function

$$\gamma_H := e^{\delta_B} \gamma_0 = F_i \psi^i + \frac{1}{3!} H_{ijk} \psi^i \psi^j \psi^k, \quad (2.53)$$

which gives the twisted Courant bracket as the antisymmetrization of (2.43).

It is evident from the general construction that Courant algebroids are uniquely encoded (up to isomorphism) in the corresponding AKSZ topological membrane theories, which are called Courant sigma-models [22]. In the particular example of the standard Courant algebroid on $E = TM \oplus T^*M$ twisted by a closed NS–NS three-form flux H , the mapping space \mathcal{M} of superfields supports the canonical BV symplectic structure

$$\omega = \int_{T[1]\Sigma_3} d^3 \hat{z} (\delta \mathbf{X}^i \delta \mathbf{F}_i + \delta \psi^i \delta \chi_i), \quad (2.54)$$

where the ghost number $U = 2$ superfields \mathbf{F}_i and $U = 0$ superfields \mathbf{X}^i , as well as the conjugate pairs of $U = 1$ superfields χ_i and ψ^i , contain each other's antifields respectively. The AKSZ construction leads to the action

$$\mathcal{S} = \int_{T[1]\Sigma_3} d^3 \hat{z} \left(\mathbf{F}_i D \mathbf{X}^i - \chi_i D \psi^i + \mathbf{F}_i \psi^i + \frac{1}{3!} H_{ijk} \psi^i \psi^j \psi^k \right), \quad (2.55)$$

which solves the classical master equation $(\mathcal{S}, \mathcal{S})_{\text{BV}} = 0$. Integrating over θ^μ and restricting to degree 0 fields in (2.55) recovers the classical action (2.39).

2.5 Lie algebroids up to homotopy and topological threebranes

Just as it proves useful to view closed strings as modes of membranes when deforming their target spaces by fluxes, it is likewise useful to view membranes as modes of higher degrees of freedom, threebranes, particularly when the membranes are regarded as the fundamental objects in M-theory with background four-form fluxes $G = dC$. With notation as previously, the threebrane theory is defined on a four-dimensional worldvolume Σ_4 , and the topological part of the bosonic action is

$$I_{\Sigma_4, G} = \int_{\Sigma_4} X^*(G) = \frac{1}{4!} \int_{\Sigma_4} G_{ijkl} dX^i \wedge dX^j \wedge dX^k \wedge dX^l . \quad (2.56)$$

This action is classically equivalent to the first order threebrane sigma-model action

$$I_{\Sigma_4, G}^{(1)} = \int_{\Sigma_4} \left(F_i \wedge (\psi^i - dX^i) + \chi_i \wedge d\psi^i + \frac{1}{4!} G_{ijkl} \psi^i \wedge \psi^j \wedge \psi^k \wedge \psi^l \right) , \quad (2.57)$$

where $\psi \in \Omega^1(\Sigma_4, X^*TM)$ and $\chi \in \Omega^2(\Sigma_4, X^*T^*M)$, while $F \in \Omega^3(\Sigma_4, X^*T^*M)$ is an auxiliary three-form. In dimension $d = 4$, the target superspace of the AKSZ construction is a QP-manifold of degree 3, which is equivalent to a higher algebroid structure introduced in [39] that arises from a homotopy deformation of a Lie algebroid. It is called a Lie algebroid up to homotopy.

Let E_0 be a vector bundle over a manifold M . We consider a general QP-manifold of degree 3 on $\mathcal{M} = T^*[3]E_0[1]$, regarded as a symplectic Lie 3-algebroid on E_0 with underlying N-manifold

$$\mathcal{M} = M \longleftarrow E_0[1] \longleftarrow E_0[1] \oplus E_0^*[2] \longleftarrow T^*[3]E_0[1] . \quad (2.58)$$

The local coordinates on \mathcal{M} are denoted $(X^i, \psi^a, \chi_a, F_i)$ with degrees $(0, 1, 2, 3)$, where X^i are local coordinates on M , ψ^a are local fiber coordinates of the shifted vector bundle $E_0[1]$, χ_a are dual fiber coordinates of $T^*[3]E_0[1] \rightarrow E_0[1]$, and F_i are local fiber coordinates of the shifted cotangent bundle $T^*[3]M$. The canonical symplectic structure is given by

$$\omega = dF_i \wedge dX^i + d\psi^a \wedge d\chi_a . \quad (2.59)$$

The most general form of a degree 4 Hamiltonian function γ on \mathcal{M} is given by a sum

$$\gamma = \gamma_k + \gamma_\rho + \gamma_T , \quad (2.60)$$

where

$$\begin{aligned} \gamma_k &= \frac{1}{2} k^{ab}(X) \chi_a \chi_b , \\ \gamma_\rho &= \rho^i_a(X) F_i \psi^a + \frac{1}{2} f^a_{bc}(X) \chi_a \psi^b \psi^c , \\ \gamma_T &= \frac{1}{4!} T_{abcd}(X) \psi^a \psi^b \psi^c \psi^d , \end{aligned} \quad (2.61)$$

are given by functions k^{ab} , ρ^i_a , f^a_{bc} and T_{abcd} on M . A Lie algebroid up to homotopy is defined with respect to this decomposition of the Hamiltonian function as the vector bundle E_0 over M with a symmetric pairing $\langle \cdot, \cdot \rangle$ on sections of E_0^* , an anchor map $\rho : E_0 \rightarrow TM$, an antisymmetric bracket $[\cdot, \cdot]_{\text{uth}}$ on sections of E_0 , a de Rham-type differential d on sections of $\bigwedge^\bullet E_0$, and a four-form Ω on E_0 . We can identify sections e of E_0 with degree 2 functions $e = f^a(X) \chi_a$ and sections α of E_0^* with degree 1 functions $\alpha = g_a(X) \psi^a$, where f^a and g_a are degree 0 functions on M . Then

the five operations are defined via derived brackets as

$$\begin{aligned}\langle \alpha_1, \alpha_2 \rangle &= \{ \{ \gamma_k, \alpha_1 \}, \alpha_2 \} , \\ \rho(e) &= \{ \{ \gamma_\rho, e \}, \cdot \} , \\ [e_1, e_2]_{\text{uth}} &= \{ \{ \gamma_\rho, e_1 \}, e_2 \} , \\ \mathfrak{d} &= \{ \gamma_\rho, \cdot \} ,\end{aligned}\tag{2.62}$$

$$\Omega(e_1, e_2, e_3, e_4) = \{ \{ \{ \{ \gamma_T, e_1 \}, e_2 \}, e_3 \}, e_4 \} .$$

The pairing additionally defines a symmetric bundle map $\mathfrak{d} : E^* \rightarrow E$ by

$$\mathfrak{d}\alpha = - \{ \gamma_k, \alpha \} .\tag{2.63}$$

The classical master equation $\{ \gamma, \gamma \} = 0$ implies that these operations obey the identities

$$\begin{aligned}[e_1, f e_2]_{\text{uth}} &= f [e_1, e_2]_{\text{uth}} + (\rho(e_1)f) e_2 \quad \text{for } f \in C^\infty(M) , \\ [[e_1, e_2]_{\text{uth}}, e_3]_{\text{uth}} + \text{cyclic} &= \mathfrak{d} \Omega(e_1, e_2, e_3, \cdot) , \\ \rho \circ \mathfrak{d} &= 0 , \\ \rho(e) \langle \alpha_1, \alpha_2 \rangle &= \langle \mathcal{L}_e \alpha_1, \alpha_2 \rangle + \langle \alpha_1, \mathcal{L}_e \alpha_2 \rangle \quad \text{with } \mathcal{L}_e := \{ \{ \gamma_\rho, e \}, \cdot \} , \\ \mathfrak{d} \circ \Omega &= 0 ,\end{aligned}\tag{2.64}$$

and we also note that $\mathfrak{d}^2 \neq 0$ in general. In other words, a symplectic Lie 3-algebroid is a vector bundle with operations $([\cdot, \cdot]_{\text{uth}}, \rho, \mathfrak{d}, \Omega)$ characterized by the algebraic identities (2.64). A particularly interesting feature behind the algebraic structure of a Lie algebroid up to homotopy is that its bracket can be extended to all degree 2 functions on $\mathcal{M} = T^*[3]E_0[1]$, which are identified as the sections of $E := E_0 \oplus \bigwedge^2 E_0^*$. This leads to a higher analogue of the Courant bracket

$$[\cdot, \cdot]_{2C} = \{ \{ \gamma, \cdot \}, \cdot \} ,\tag{2.65}$$

where now the full Hamiltonian function is used. We shall call it a 2-Courant bracket in the following.

The simplest relevant example for us is what we shall call the *standard* Lie algebroid up to homotopy, which is the case $E_0 = TM$. The symplectic structure is

$$\omega = dX^i \wedge dF_i + d\psi^i \wedge d\chi_i .\tag{2.66}$$

We choose the Liouville potential given by $\vartheta = F_i dX^i + \chi_i d\psi^i$. The simplest Hamiltonian function from (2.60) and (2.61) has identity anchor map $\rho^i_j = \delta^i_j$ with all other structure functions equal to zero, and is given by

$$\gamma_0 = F_i \psi^i .\tag{2.67}$$

The cohomological vector field is again the de Rham vector field $Q_{\gamma_0} = \psi^i \frac{\partial}{\partial X^i}$ on $\mathcal{M} = T^*[3]T[1]M$. In this instance, the derived bracket on degree 2 functions

$$A^i \chi_i + \frac{1}{2} \lambda_{ij} \psi^i \psi^j \longleftrightarrow A^i \frac{\partial}{\partial X^i} + \frac{1}{2} \lambda_{ij} dX^i \wedge dX^j\tag{2.68}$$

gives the standard 2-Courant bracket on the vector bundle

$$E = TM \oplus \bigwedge^2 T^*M ,\tag{2.69}$$

which reads explicitly as

$$[A + \lambda, B + \xi]_{2C} = [A, B] + \mathcal{L}_A \xi - \mathcal{L}_B \lambda + \frac{1}{2} d(\iota_B \lambda - \iota_A \xi)\tag{2.70}$$

for vector fields A, B and two-forms λ, ξ on M .⁶ The standard 2-Courant bracket (2.70) appears in

⁶This is called a Vinogradov algebroid in [11, 40, 41].

exceptional generalized geometry as the natural bracket which is compatible with the commutator algebra of generalized Lie derivatives [42, 43].

One can also introduce a flux deformation by an additional term γ_T in the Hamiltonian function, which twists the standard 2-Courant bracket by a four-form which is necessarily closed by the classical master equation. Given an M-theory three-form C -field on M , with four-form flux $G = dC$, canonical transformation of the Hamiltonian function γ_0 by the degree 3 function $C = \frac{1}{3!} C_{ijk}(X) \psi^i \psi^j \psi^k$ on \mathcal{M} yields the twisted Hamiltonian function

$$\gamma_G := e^{\delta_C} \gamma_0 = F_i \psi^i + \frac{1}{4!} G_{ijkl} \psi^i \psi^j \psi^k \psi^l, \quad (2.71)$$

and it gives the twisted standard 2-Courant bracket as

$$[A + \lambda, B + \xi]_{2C, G} = [A, B] + \mathcal{L}_A \xi - \mathcal{L}_B \lambda + \frac{1}{2} d(\iota_B \lambda - \iota_A \xi) + \iota_A \iota_B G. \quad (2.72)$$

One can now use the AKSZ construction to build BV quantized sigma-models in four dimensions based on degree 3 QP-manifolds, which we shall call 2-Courant sigma-models. For the standard Lie algebroid up to homotopy on $E_0 = TM$ twisted by a closed four-form flux G , the BV bracket is

$$(\cdot, \cdot)_{BV} = \int_{T[1]\Sigma_4} d^4 \hat{z} \left(\frac{\delta}{\delta \mathbf{X}^i} \wedge \frac{\delta}{\delta \mathbf{F}_i} + \frac{\delta}{\delta \chi_i} \wedge \frac{\delta}{\delta \psi^i} \right), \quad (2.73)$$

and the classical master equation $(\mathbf{S}, \mathbf{S})_{BV} = 0$ is solved by the topological threebrane action

$$\mathbf{S} = \int_{T[1]\Sigma_4} d^4 \hat{z} \left(\mathbf{F}_i D\mathbf{X}^i + \psi^i D\chi_i + \mathbf{F}_i \psi^i + \frac{1}{4!} G_{ijkl} \psi^i \psi^j \psi^k \psi^l \right). \quad (2.74)$$

Integrating over θ^μ and restricting to degree 0 fields in (2.74) recovers the classical action (2.57).

2.6 Dimensional reduction and effective actions

In this paper we shall also derive some novel relations amongst AKSZ sigma-models in the various dimensions d through a procedure of dimensional reduction. For this, we follow [44] where a practical dimensional reduction method, called ‘Losev’s trick’, is employed. Let us briefly recall the main ingredients, which are rooted in the construction of effective actions in the BV formalism.

The symplectic structure ω on the target supermanifold \mathcal{M} induces a natural second order differential operator Δ , which in local coordinates can be given as

$$\Delta = \frac{1}{2} \omega^{\hat{i}\hat{j}} \frac{\vec{\partial}^2}{\partial \hat{X}^{\hat{i}} \partial \hat{X}^{\hat{j}}}, \quad (2.75)$$

where $\omega^{\hat{i}\hat{j}}$ is the inverse of $\omega_{\hat{i}\hat{j}}$. This pulls back to give the BV operator Δ for the BV bracket $(\cdot, \cdot)_{BV}$ on the space of AKSZ fields \mathcal{M} ; it has ghost degree 1 and satisfies $\Delta^2 = 0$. Since $\Delta \mathbf{S} = 0$, the AKSZ action \mathbf{S} satisfies the BV quantum master equation $\Delta e^{-\mathbf{S}/\hbar} = 0$ on \mathcal{M} , which is equivalent to

$$\frac{1}{2} (\mathbf{S}, \mathbf{S})_{BV} = \hbar \Delta \mathbf{S}, \quad (2.76)$$

and follows from nilpotency of the quantum version of the cohomological vector field $\mathbf{Q} - \hbar \Delta$. This ensures independence of the BRST-invariant quantum field theory on the choice of gauge fixing, provided we define the path integral by additionally equipping \mathcal{M} with a measure μ which is compatible with ω [34].

Borrowing standard terminology from renormalization of quantum field theory, let us now assume that the space of AKSZ fields can be decomposed into a direct product

$$\mathcal{M} = \mathcal{M}_{\text{UV}} \times \mathcal{M}_{\text{IR}} \quad (2.77)$$

of ultraviolet (UV) and infrared (IR) degrees of freedom, with a compatible decomposition of the canonical symplectic form $\omega = \omega_{\text{UV}} + \omega_{\text{IR}}$, where ω_{UV} is a BV symplectic structure on \mathcal{M}_{UV} and ω_{IR} is a BV symplectic structure on \mathcal{M}_{IR} . Then the BV Laplacian also decomposes as $\Delta = \Delta_{\text{UV}} + \Delta_{\text{IR}}$. One now ‘integrates out’ the ultraviolet degrees of freedom to get an effective action. The integration requires a gauge fixing on the ultraviolet sector \mathcal{M}_{UV} of the space of superfields, which means a choice of a Lagrangian submanifold $\mathcal{L} \hookrightarrow \mathcal{M}_{\text{UV}}$. Then the effective BV action S_{eff} in the infrared sector is defined as

$$e^{-S_{\text{eff}}/\hbar} := \int_{\mathcal{L}} \sqrt{\mu}_{\mathcal{L}} e^{-S/\hbar}, \quad (2.78)$$

where $\sqrt{\mu}_{\mathcal{L}}$ is the measure on \mathcal{L} induced by μ . Then the effective action satisfies the quantum master equation $\Delta_{\text{IR}} e^{-S_{\text{eff}}/\hbar} = 0$. A change of gauge fixing in the ultraviolet sector, corresponding to a deformation of the Lagrangian submanifold \mathcal{L} , only changes $e^{-S_{\text{eff}}/\hbar}$ by a Δ_{IR} -exact term. Similarly, the value of the partition function is independent of the particular choice of splitting (2.77) by the Batalin-Vilkovisky theorem [34].

Formally, this technique defines a pushforward by the projection map $\mathcal{M} \rightarrow \mathcal{M}_{\text{IR}}$ onto the infrared sector of the space of superfields. In the following we use this method to reduce our AKSZ sigma-model actions to AKSZ theories in lower dimensions.

2.7 Double dimensional reduction of twisted sigma-models

As a simple example of the dimensional reduction technique described in §2.6 above, we describe the reduction of the AKSZ threebrane sigma-model to the AKSZ membrane sigma-model with flux deformations. To motivate the reduction of threebrane flux to membrane flux, consider the simple topological threebrane action (2.56) given by the pullback of a closed four-form flux G on a d -dimensional manifold M by the worldvolume map $X : \Sigma_4 \rightarrow M$. We perform a double dimensional reduction on a circle taking both the worldvolume and the target to be product manifolds $\Sigma_4 = \Sigma_3 \times S^1$ and $M = \widetilde{M} \times S^1$, with \widetilde{M} a manifold of dimension $d - 1$, and wrap the S^1 of the worldvolume around the S^1 of the target space; in other words, we regard the membranes as modes of threebranes wrapping S^1 . We write the local coordinates on the worldvolume Σ_4 as $\sigma = (\tilde{\sigma}, t)$, where $\tilde{\sigma} \in \Sigma_3$ and t is the coordinate on S^1 . The target space coordinate indices are $I = (i, d)$, where $i = 1, \dots, d - 1$ label directions along \widetilde{M} .

Wrapping the target circle means that the map X has the local expression

$$X = (X^I(\sigma)) = (\tilde{X}^i(\tilde{\sigma}), w t) \quad (2.79)$$

with the reduced map $\tilde{X} : \Sigma_3 \rightarrow \widetilde{M}$ and $X^d = w t$, where w is a winding number. The dimensional reduction of the action $I_{\Sigma_4, G}$ from (2.56) is then given by $I_{\Sigma_3, H}$ from (2.38), where the closed three-form flux H on \widetilde{M} is given by

$$H_{ijk}(\tilde{X}) = w \int_{S^1} dt G_{ijkd}(\tilde{X}, t). \quad (2.80)$$

Hence the threebrane flux G reduces to a membrane flux H under double dimensional reduction on a circle. We shall now show that this reduction also works at the level of the full AKSZ sigma-models.

We start with the G -twisted standard 2-Courant sigma-model given by (2.74), and use the dimensional reduction method of §2.6. We write the expansion of an arbitrary superfield $\phi \in \mathcal{M}$ with respect to the coordinate direction t as

$$\phi = \tilde{\phi} + \phi_t \theta^t, \quad (2.81)$$

where neither $\tilde{\phi}$ nor ϕ_t contain the odd coordinate θ^t . If ϕ has ghost number n , then $\tilde{\phi}$ has ghost number n and ϕ_t has ghost number $n - 1$. We choose the infrared fields to be $(\mathbf{F}_t)_i$, $\tilde{\mathbf{X}}^i$, $\tilde{\psi}^i$ and $(\chi_t)_i$. On the ultraviolet fields we fix the gauge by choosing the Lagrangian submanifold \mathcal{L} defined by

$$\mathbf{X}_t^I = 0, \quad \tilde{\mathbf{X}}^d = -w t, \quad \psi_t^i = 0, \quad \psi_t^d = w \quad \text{and} \quad \tilde{\psi}^d = 0. \quad (2.82)$$

The equations of motion for $\tilde{\mathbf{F}}_i$ and $\tilde{\chi}_i$ give $\partial_t \tilde{\mathbf{X}}^i = 0$ and $\partial_t \tilde{\psi}^i = 0$, and in this way we get the AKSZ action of the H -twisted standard Courant sigma-model (2.55) and its BV symplectic form (2.54) with the definitions of the fields

$$\mathbf{X}^i = \tilde{\mathbf{X}}^i, \quad \mathbf{F}_i = \int_{S^1} dt (\mathbf{F}_t)_i, \quad \psi^i = \tilde{\psi}^i \quad \text{and} \quad \chi_i = \int_{S^1} dt (\chi_t)_i, \quad (2.83)$$

and H -flux as in (2.80). We refer to this type of gauge fixing as a double dimensional reduction on a circle.

It is worth stressing that the kinetic terms are necessary in this construction because without the term $\tilde{\mathbf{F}}_d \partial_t \tilde{\mathbf{X}}^d$, the term $\tilde{\mathbf{F}}_d \psi_t^d$ gives $w \tilde{\mathbf{F}}_d$, which would yield $w = 0$ and vanishing H -flux on-shell. An interesting feature here is that the term coming from the Liouville potential $\psi^i d\chi_i$ of the threebrane has been reversed via the reduction to the Liouville potential $-\chi_i d\psi^i$ of the membrane. Note also that this dimensional reduction can be done at the purely bosonic level without the ghost fields: Starting from (2.57), we use the expression (2.79) for the wrapping of X , and then the equations of motion for the three-form field F_I and reduced two-form fields gives the bosonic part of the standard Courant sigma-model with H -flux in (2.39).

By a direct computation in local coordinates, it is further possible to show that the standard 2-Courant bracket (2.70) on $M = \tilde{M} \times S^1$ suitably reduces to the standard Courant bracket (2.52) on \tilde{M} . The dimensional reduction of the 2-Courant sigma-model to the Courant sigma-model is analogous to the reduction discussed by [43] in the context of $SL(5)$ exceptional field theory, wherein the $SL(5)$ generalized Courant bracket reduces to the $O(3,3)$ generalized Courant bracket (C-bracket) of double field theory.

3 AKSZ theories of topological membranes on G_2 -manifolds

In this section we start to focus our attention on topological membrane models in M-theory which have reductions to the A-model. We study two topological membrane theories on G_2 -manifolds. We begin by reviewing the topological membrane model of [3] which is based on the Mathai-Quillen formalism,⁷ as well as the BRST model of [6]. We supplement the Mathai-Quillen construction with an auxiliary field, analogously to the construction of [6], and we give AKSZ formulations which reproduce both membrane models after gauge fixing.

3.1 Topological membrane theories

Mathai-Quillen membrane sigma-model. Let us begin by reviewing the topological membrane theory of [3], which we call the Mathai-Quillen membrane sigma-model. Let M_7 be an oriented

⁷For further details about the Mathai-Quillen formalism in general, see e.g. [4, 5].

seven-dimensional Riemannian manifold with G_2 -structure, which is equivalent to equipping M_7 with a global three-form Φ that is closed, $d\Phi = 0$, and coclosed, $d*\Phi = 0$, where $*$ is the Hodge duality operator with respect to the metric g of M_7 . Given an embedding map $X : \Sigma_3 \rightarrow M_7$, let us introduce a local section of the cotangent bundle T^*M_7 by

$$\Xi_I = \frac{1}{3!} (*\Phi)_{IJKL} \partial_\mu X^J \partial_\nu X^K \partial_\rho X^L \epsilon^{\mu\nu\rho} , \quad (3.1)$$

where Greek indices label local coordinates σ^μ on the worldvolume Σ_3 , with $\partial_\mu := \frac{\partial}{\partial \sigma^\mu}$, and capital Latin indices label coordinates X^I on M_7 , with $\partial_I := \frac{\partial}{\partial X^I}$. The symbol $\epsilon^{\mu\nu\rho}$ is the Levi-Civita tensor density on Σ_3 . If Ξ_I vanishes, then $X(\Sigma_3) \subset M_7$ is called an associative three-cycle.

We further introduce a ghost field ψ^I on Σ_3 with ghost number 1 and an antighost field χ^I on Σ_3 with ghost number -1 . Then the action of the Mathai-Quillen membrane sigma-model is

$$I_{\text{MQ}} = \int_{\Sigma_3} d^3\sigma \left(\frac{1}{2} g^{IJ} \Xi_I \Xi_J + i \chi^I (\delta \Xi_I - \Gamma^K_{IJ} \psi^J \Xi_K) - \frac{1}{4} R_{IJKL} \psi^I \psi^J \chi^K \chi^L \right) , \quad (3.2)$$

where

$$\delta \Xi_I - \Gamma^K_{IJ} \psi^J \Xi_K = \frac{1}{2} (*\Phi)_{IJKL} \nabla_\mu \psi^J \partial_\nu X^K \partial_\rho X^L \epsilon^{\mu\nu\rho} , \quad (3.3)$$

with $\nabla_\mu \psi^I = \partial_\mu \psi^I + \Gamma^I_{JK} \psi^J \partial_\mu X^K$ given by the Levi-Civita connection of the metric g pulled back to Σ_3 by X , and R^I_{JKL} are the components of the Riemann curvature tensor of g .⁸ The action (3.2) is invariant under the BRST transformations

$$\delta X^I = \psi^I , \quad \delta \psi^I = 0 \quad \text{and} \quad \delta \chi^I = i g^{IJ} \Xi_J - \Gamma^I_{JK} \psi^J \chi^K , \quad (3.4)$$

which is nilpotent only on-shell, and it is BRST-exact up to the equations of motion:

$$I_{\text{MQ}} = \delta \Psi'_{\text{MQ}} \quad \text{with} \quad \Psi'_{\text{MQ}} = -\frac{i}{2} \int_{\Sigma_3} d^3\sigma \chi^I \Xi_I . \quad (3.5)$$

The fixed point locus of the BRST charge is the space of associative three-cycles $X : \Sigma_3 \rightarrow M_7$, which are membrane instantons.

Let us now linearize the BRST transformations by supplementing the Mathai-Quillen membrane sigma-model with an auxiliary field. We define an auxiliary field b^I with the new BRST transformations

$$\delta X^I = \psi^I , \quad \delta \psi^I = 0 , \quad \delta \chi^I = b^I \quad \text{and} \quad \delta b^I = 0 , \quad (3.6)$$

which is now nilpotent off-shell, and the membrane action is BRST-exact with the gauge fixing fermion

$$\Psi_{\text{MQ}} = - \int_{\Sigma_3} d^3\sigma \chi^I \left(i \Xi_I + \frac{1}{2} \Gamma_{IJK} \chi^J \psi^K - \frac{1}{2} g_{IJ} b^J \right) . \quad (3.7)$$

Then the membrane action $S_{\text{MQ}} = \delta \Psi_{\text{MQ}}$ is given by

$$S_{\text{MQ}} = \int_{\Sigma_3} d^3\sigma \left(-i b^I \Xi_I + \chi^I (i \delta \Xi_I + \Gamma_{IJK} b^J \psi^K) + \frac{1}{2} \partial_L \Gamma_{IJK} \chi^I \chi^J \psi^K \psi^L + \frac{1}{2} g_{IJ} b^I b^J \right) . \quad (3.8)$$

The equation of motion for b^I gives

$$b^I = i g^{IJ} \Xi_J - \Gamma^I_{JK} \chi^J \psi^K . \quad (3.9)$$

Using this expression one can show that the membrane action (3.8) reduces to the Mathai-Quillen membrane action (3.2).

⁸Capital Latin indices are raised and lowered with the metric g .

BTZ membrane sigma-model. In [6] a different topological membrane action on G_2 -manifolds is given, which is based on BRST quantization of the topological action $I_{\Sigma_3, \Phi} = \int_{\Sigma_3} X^*(\Phi)$; we call it the Bonelli-Tanzini-Zabzine (BTZ for short) membrane sigma-model. With the same fields and notation as above, the action is

$$S_{\text{BTZ}} = -I_{\Sigma_3, \Phi} + \delta\Psi_{\text{BTZ}} , \quad (3.10)$$

with the gauge fixing fermion

$$\Psi_{\text{BTZ}} = \int_{\Sigma_3} d^3\sigma \chi^I \left(g_{IJ} \dot{X}^J + \Phi_{IJK} \partial_1 X^J \partial_2 X^K + \frac{1}{2} \Gamma_{IJK} \chi^J \psi^K - \frac{1}{2} g_{IJ} b^J \right) , \quad (3.11)$$

where the worldvolume indices run through $\mu = 0, 1, 2$ and the dot denotes the action of the derivative ∂_0 . The BRST transformations are the same as those of the Mathai-Quillen membrane model in (3.6), thus they have identical BV formulations. Since $d\Phi = 0$, the topological flux term $I_{\Sigma_3, \Phi}$ in the AKSZ framework arises from a canonical transformation as in §2.4, and consequently it has no effect in the BV algebra on the mapping space \mathcal{M} . Hence in the following we will only study the BRST-exact term in (3.10).

3.2 BV formulation and AKSZ constructions

Both topological membrane sigma-models are described by a gauge fixing fermion $\Psi[X^I, \psi^I, \chi^I, b^I]$. The only non-zero BRST transformations are $\delta X^I = \psi^I$ and $\delta \chi^I = b^I$, so

$$\delta\Psi = \int_{\Sigma_3} d^3\sigma \left(\psi^I \frac{\delta\Psi}{\delta X^I} + b^I \frac{\vec{\delta}\Psi}{\delta \chi^I} \right) . \quad (3.12)$$

With the definition of the antifields⁹

$$X_I^+ = \frac{\delta\Psi}{\delta X^I} \quad \text{and} \quad \chi_I^+ = \frac{\vec{\delta}\Psi}{\delta \chi^I} , \quad (3.13)$$

we can rewrite the BRST-exact part of the membrane actions as

$$\delta\Psi = \int_{\Sigma_3} d^3\sigma \left(\psi^I X_I^+ + b^I \chi_I^+ \right) . \quad (3.14)$$

Thus the BRST-exact membrane actions in (3.8) and (3.10) differ only in the choice of gauge fixing, i.e. in the choice of Lagrangian submanifold $\mathcal{L} \subset \mathcal{M}$. In the following we propose two different AKSZ constructions for these topological membrane theories.

AKSZ construction I. Our first AKSZ construction contains a rather large number of fields, but very few of them are explicitly used in the gauge fixed action. The source dg-manifold is $\mathcal{W} = T[1]\Sigma_3$ as usual, and the target symplectic dg-manifold is $\mathcal{M} = T^*[2]T[-1]T[1]M_7$. The base coordinates in $T[-1]T[1]M_7$ are $(X^I, \xi^I, B^I, \eta^I)$ with degree $(0, 1, 0, -1)$, where X^I are associated to the coordinates of M_7 . The graded fiber coordinates are (F_I, ζ_I, N_I, G_I) with degree $(2, 1, 2, 3)$, and the canonical symplectic structure of degree 2 on \mathcal{M} is

$$\omega_{3,1} = dF_I \wedge dX^I + d\zeta_I \wedge d\xi^I + dN_I \wedge dB^I + dG_I \wedge d\eta^I . \quad (3.15)$$

In the following we expand a general AKSZ superfield $\phi \in \mathcal{M}$ as in (2.23) for $d = 3$. Our membrane BRST fields X^I, ψ^I, χ^I, b^I do not have form components, so we choose them as the zeroth or third

⁹As in §2.2 we denote the antifield of a field ϕ by ϕ^+ .

components of a superfield. Our choice in this first construction is as the zeroth component for both membrane models, and their antifields are assigned to the third components. Explicitly this means we take

$$\begin{aligned}
X^{(0)I} &= X^I & \text{and} & & F_I^{(3)} &= X_I^+ , \\
\xi^{(0)I} &= \psi^I & \text{and} & & \zeta_I^{(3)} &= \psi_I^+ , \\
\eta^{(0)I} &= \chi^I & \text{and} & & G_I^{(3)} &= \chi_I^+ , \\
B^{(0)I} &= b^I & \text{and} & & N_I^{(3)} &= b_I^+ .
\end{aligned} \tag{3.16}$$

The AKSZ action is constructed without kinetic terms and with a degree 3 Hamiltonian function γ such that the corresponding BV bracket with the associated cohomological vector field \mathbf{Q} on \mathcal{M} generates the BRST transformations (3.6). Thus we take

$$S_{G_2, I} = \int_{T[1]\Sigma_3} d^3\hat{z} \left(\xi^I \mathbf{F}_I + \mathbf{B}^I \mathbf{G}_I \right) , \tag{3.17}$$

which has eight components after the expanding the superfields. We use a gauge fixing fermion to set the antifields X_I^+ , ψ_I^+ , χ_I^+ , b_I^+ , and we choose the gauge fixing of (3.17) on the other fields to give the gauge fixed action (3.14). For example, we may choose the Lagrangian submanifold \mathcal{L} determined by the equations

$$\begin{aligned}
X^{(1)I} &= X^{(3)I} = 0 & \text{and} & & F_I^{(1)} &= 0 , \\
\xi^{(1)I} &= \xi^{(3)I} = 0 & \text{and} & & \zeta_I^{(1)} &= 0 , \\
\eta^{(1)I} &= \eta^{(3)I} = 0 & \text{and} & & G_I^{(1)} &= 0 , \\
B^{(1)I} &= B^{(3)I} = 0 & \text{and} & & N_I^{(1)} &= 0 ,
\end{aligned} \tag{3.18}$$

for the antifields. The other antifields given by the gauge fixing fermion are X_I^+ , χ_I^+ , ψ_I^+ and b_I^+ . If we choose (3.7) we get the Mathai-Quillen membrane action (3.8), while if we choose (3.11) we get the BRST-exact part of the BTZ topological membrane action (3.10).

For example, in the Mathai-Quillen membrane sigma-model the pertinent antifields are given by

$$\begin{aligned}
X_I^+ &= \frac{\delta \Psi_{\text{MQ}}}{\delta X^I} = -i \frac{\delta}{\delta X^I} \int_{\Sigma_3} d^3\sigma \chi^I \Xi_I - \frac{1}{2} \partial_I \Gamma_{JKL} \chi^J \chi^K \chi^L + \frac{1}{2} \partial_I g_{JK} \chi^J b^K , \\
\chi_I^+ &= \frac{\overrightarrow{\delta} \Psi_{\text{MQ}}}{\delta \chi^I} = -i \Xi_I - \Gamma_{[IJ]K} \chi^J \psi^K + \frac{1}{2} g_{IJ} b^J ,
\end{aligned} \tag{3.19}$$

and it is easy to see

$$\psi^I \frac{\delta}{\delta X^I} \int_{\Sigma_3} d^3\sigma \chi^I \Xi_I = -\chi^I \delta \Xi_I , \tag{3.20}$$

so that gauge fixing the antifields in this way restricts the AKSZ action functional (3.17) on \mathcal{L} to the action (3.8). The gauge fixing with Ψ_{BTZ} is very similar, and it gives the BTZ membrane action (3.10). Note that it is possible to add kinetic terms to the AKSZ action, and then set them to zero with a more specific gauge fixing choice, but evidently the model (3.17) is simpler to work with.

AKSZ construction II. We introduce another AKSZ construction for both topological membrane theories, which is based on the standard Courant sigma-model from §2.4. The BV action that

we want to reproduce in the AKSZ theory is again (3.14), but now we define the fermionic fields ψ^I and χ^I as one-forms in the superfield formalism. The target in this case is taken to be the QP-manifold $\mathcal{M} = T^*[2]T[1]M_7$ of degree 2 corresponding to the standard Courant algebroid on $TM_7 \oplus T^*M_7$, which contains half as many coordinates compared to the previous construction. The notation for the coordinates are the same as before, so that $(X^I, F_I, \xi^I, \zeta_I)$ have degrees $(0, 2, 1, 1)$. The symplectic form is

$$\omega_{3,\text{II}} = dF_I \wedge dX^I + d\zeta_I \wedge d\xi^I . \quad (3.21)$$

The relevant fields in the superfield formalism are

$$\begin{aligned} X^{(0)I} &= X^I & \text{and} & & X_0^{(1)I} &= \chi^I , \\ (F_I^{(2)})_{12} &= \chi_I^+ & \text{and} & & (F_I^{(3)})_{012} &= X_I^+ , \\ \xi^{(0)I} &= \psi^I & \text{and} & & \xi_0^{(1)I} &= b^I , \\ (\zeta_I^{(2)})_{12} &= -b_I^+ & \text{and} & & (\zeta_I^{(3)})_{012} &= \psi_I^+ , \end{aligned} \quad (3.22)$$

where we used an explicit worldvolume index convention to define the membrane fields χ^I , b^I and their antifields. The BV action then simply corresponds to the untwisted Hamiltonian function γ_0 from §2.4 and reads

$$\mathcal{S}_{G_2,\text{II}} = \int_{T[1]\Sigma_3} d^3\hat{z} \, \boldsymbol{\xi}^I \mathbf{F}_I . \quad (3.23)$$

There are many possible gauge fixings which recover the action (3.14). One choice is to take the Lagrangian submanifold defined by

$$\begin{aligned} F_I^{(0)} &= F_I^{(1)} = 0 & \text{and} & & \zeta_I^{(0)} &= \zeta_I^{(1)} = 0 , \\ (F_I^{(2)})_{01} &= 0 & \text{and} & & (\zeta_I^{(2)})_{01} &= 0 , \\ (F_I^{(2)})_{13} &= 0 & \text{and} & & (\zeta_I^{(2)})_3 &= 0 . \end{aligned} \quad (3.24)$$

The residual antifields are again set by the gauge fixing fermion $\Psi[X^I, \psi^I, \chi^I, b^I]$, given in (3.7) for the Mathai-Quillen membrane sigma-model and in (3.11) for the BTZ membrane sigma-model.

It is an interesting feature of our first AKSZ construction that the two terms in (3.17) are decoupled from each other, in the sense that they can be gauge fixed separately and decoupled in the AKSZ action as well. This means that one can remove the second term with a gauge fixing to get our second AKSZ constructions, but they differ from those proposed for the topological membranes, because the antifields are assigned differently.

3.3 Derived brackets

The main geometric distinction between the two AKSZ membrane theories we have constructed above is that the second construction is based on a target which is a QP-manifold of degree 2, corresponding to the standard Courant algebroid, whereas the first construction is based on a target which is not an N-manifold, as it involves local affine coordinates of degree -1 , and consequently does not correspond to a symplectic Lie 2-algebroid. Passing to dg-manifolds which are equipped with negative gradings is of course natural and standard in the BV–BRST formalism, wherein ghost fields and antifields typically come with negative gradings, but it takes us out of the realm of graded geometry into derived geometry [45]: Whereas non-negatively graded symplectic dg-manifolds generally correspond to symplectic L_∞ -algebroids, those which are arbitrarily graded correspond to

derived symplectic L_∞ -algebroids. The relevance of L_∞ -algebroids in BV quantization was already emphasised by [7, 46], but entering into further discussion of these geometric facts would take us far away from the scope of the present paper, so we content ourselves in pointing out a few interesting geometric consequences of the corresponding derived bracket construction.

The degree 3 Hamiltonian function on $\mathcal{M} = T^*[2]T[-1]T[1]M_7$ corresponding to the first AKSZ action (3.17) is given by

$$\gamma_{G_2, I} = F_I \xi^I + G_I B^I. \quad (3.25)$$

Its first term is the same as the Hamiltonian function (2.49) for the standard Courant algebroid, so its derived brackets gives the standard Courant bracket (2.52) on degree 1 functions of (X, ξ, ζ) . Moreover, this is also the derived bracket of the Hamiltonian function corresponding to the second AKSZ action (3.23), which contains solely the first term of (3.25).

The interesting feature here is the consequence of the second term in (3.25) and the negative degree coordinates η^I . The derived bracket of a symplectic dg-manifold with symplectic structure of degree 2 is defined on degree 1 functions. Such a function f can be expanded in the form

$$f = f^{(0)}(X, B, \xi, \zeta) + f_I^{(1)}(X, B, \xi, \zeta, F, N) \eta^I + \sum_{l=2}^7 f_{I_1 \dots I_l}^{(l)}(X, B, \xi, \zeta, F, N, G) \eta^{I_1} \dots \eta^{I_l}, \quad (3.26)$$

where $f^{(l)}$ is an l -form in the non-negatively graded coordinates on \mathcal{M} of degree $l+1$. The second term $G_I B^I$ in the Hamiltonian function decouples on the zeroth order functions $f^{(0)}(X, B, \xi, \phi)$, since it does not contain any of the canonically conjugate coordinates to X, B, ξ or ζ . Hence our derived bracket is closed on the subspace of zeroth order functions $f^{(0)}$, where it gives the standard Courant bracket (2.52), with the coefficients now depending on the two degree 0 coordinates X and B . The degree 0 fields are doubled in this sense, but they play an asymmetric role in the underlying geometric structure.

The restriction of the derived bracket to any higher order in η^I is no longer closed, and only closes if we consider all orders at once. Thus our derived bracket appears as an infinite extension of the standard Courant bracket, which contains the standard Courant bracket as the subalgebra of functions which are independent of η^I . This structure underlies the derived symplectic L_∞ -algebroid over M_7 alluded to above.¹⁰

3.4 Dimensional reductions from topological threebrane theories

In §2.5 we introduced an AKSZ topological threebrane sigma-model which has the standard 2-Courant bracket as its derived algebraic structure on a graded target space which is a QP-manifold of degree 3. We can shed further light on the algebroid structure discussed in §3.3 by considering our membrane models as arising through certain reductions of such a threebrane theory. We first consider this sigma-model without a four-form flux deformation and defined for the G_2 -manifold $M = M_7$. We suppose that the threebrane worldvolume is a product manifold $\Sigma_4 = \Sigma_3 \times S^1$, and that all superfields are independent of the extra coordinate t of S^1 . Using the same notation (2.81) for the expansion of an arbitrary superfield, integration over the odd coordinate θ^t in the action (2.74) without the flux term leads to the AKSZ action¹¹

$$S_{3, \text{red}} = \int_{T[1]\Sigma_3} d^3 \hat{z} (G_I B^I + F_I \xi^I - F_I D X^I + \xi^I D \zeta_I - G_I D \eta^I - B^I D N_I) \quad (3.27)$$

¹⁰See e.g. [40] for a general definition of L_∞ -algebroids.

¹¹The kinetic part of the AKSZ action is given here by $-\vartheta$, where ϑ is the Liouville potential on the symplectic dg-manifold \mathcal{M} .

and the BV symplectic form

$$\omega_{3,\text{red}} = \int_{T[1]\Sigma_3} d^3\hat{z} \left(\delta F_I \delta X^I - \delta \zeta_I \delta \xi^I + \delta G_I \delta \eta^I - \delta N_I \delta B^I \right), \quad (3.28)$$

where we have introduced the fields

$$\begin{aligned} F_I &= -(\mathbf{F}_t)_I, & G_I &= \tilde{\mathbf{F}}_I, & \zeta_I &= (\chi_t)_I & \text{and} & N_I &= \tilde{\chi}_I, \\ X^I &= \tilde{X}^I, & \eta^I &= X_t^I, & B^I &= -\psi_t^I & \text{and} & \xi^I &= -\tilde{\psi}^I, \end{aligned} \quad (3.29)$$

and rescaled them by the length of S^1 . Thus the reduced AKSZ action without the kinetic terms is our first AKSZ membrane action (3.17), up to a few sign differences appearing in the symplectic forms which can be resolved with a redefinition of the original symplectic form of the membrane sigma-model that leaves its gauge fixed action invariant. On the other hand, the kinetic terms can be removed with the same gauge fixing that we used to obtain the topological membrane theories in this section. In this way, the threebrane AKSZ action without any kinetic term

$$\int_{T[1]\Sigma_4} d^4\hat{z} F_I \psi^I \quad (3.30)$$

is a straightforward extension of our AKSZ membrane sigma-models.

This means therefore that our first AKSZ construction for topological membranes on G_2 -manifolds is a reduced AKSZ theory of topological threebranes on the same target space. The special feature of the threebrane theory is that its derived bracket on the target QP-manifold $T^*[3]T[1]M_7$ of degree 3 gives the standard 2-Courant bracket (2.70) on the vector bundle $E = TM_7 \oplus \bigwedge^2 T^*M_7$, which relates the geometry behind our specific AKSZ construction to the exceptional generalized geometry of M-theory.

The second AKSZ construction for topological membranes from §3.2 can also be reformulated within a topological threebrane sigma-model, in the same way as the first construction. The only difference is that we get an additional term in the AKSZ action after the reduction, which can be set to zero with gauge fixing, because we do not need those fields to get the topological membrane theories with further gauge fixing. Hence the action (3.30) reduces to the second AKSZ sigma-model action as well.

In §2.7 we saw that viewing membranes as wrapping modes of threebranes, by wrapping the world-volume circle on the target circle, reduces the four-dimensional standard 2-Courant sigma-model with G -flux to the three-dimensional standard Courant sigma-model with H -flux. This means that it is possible to add G -flux to our topological membrane theories at the threebrane level. Although the reduction above, wherein the fields are taken to be independent of one worldvolume direction, removes the topological flux term in (2.56), at the level of the full AKSZ action it does not. It leaves an extra contribution

$$\frac{1}{3!} \int_{T[1]\Sigma_3} d^3\hat{z} G_{IJKL} \xi^I \xi^J \xi^K B^L, \quad (3.31)$$

which can be taken as a definition of a flux deformation for our first AKSZ membrane construction in §3.2.

Alternatively, one can directly induce the topological flux deformation $I_{\Sigma_3, \Phi}$ that we neglected in the action (3.10) by applying the double dimensional reduction technique from §2.7. For this, we first note that, generally, the AKSZ threebrane sigma-model (2.74) gives the BV action for the sigma-model of [11] for topological threebranes on an eight-dimensional $Spin(7)$ -manifold M_8 ,

with the twist G taken to be the global self-dual closed four-form corresponding to the $Spin(7)$ -structure on M_8 [39], whose path integral localizes on Cayley four-cycles (threebrane instantons). We can then embed our topological brane sigma-models with target G_2 -manifold (M_7, Φ) into this threebrane theory by taking $\Sigma_4 = \Sigma_3 \times S^1$ and $M_8 = M_7 \times S^1$ with the Cayley four-form

$$G = dX^8 \wedge \Phi + *\Phi . \quad (3.32)$$

Using double dimensional reduction on a circle as in §2.7 then reproduces the H -twisted standard Courant sigma-model (2.55) with flux $H = w\Phi$, and consequently leads to our second AKSZ construction from §3.2 with topological term. On the other hand, if the original threebrane is localized on S^1 , i.e. X^8 is constant, then the threebrane theory reduces on t -independent superfields as above to our first AKSZ construction, with extra flux term (3.31) given by $G = *\Phi$. In this setting these threebrane worldvolume theories are regarded as providing a microscopic description of topological F-theory [6, 47].

4 AKSZ theories for the topological A-model

The A- and B-models of the topological sigma-model [48] have been extensively studied over the past three decades, particularly when they are coupled to gravity where they become the A- and B-models of topological string theory. They were also one of the first examples of the AKSZ construction from [7]. In particular, all known AKSZ constructions for the A-model are Poisson sigma-models [7, 10, 17, 26, 28, 30], so they all have the same target QP-manifold of degree 1, symplectic structure and Hamiltonian function as in §2.3. The Poisson bivector π in these instances is given by the inverse of the Kähler form on the target Calabi-Yau manifold, and the AKSZ sigma-models all reduce to the A-model in particular gauges.

In this section we will follow the general procedure of §2.6 to compute a dimensional reduction, at the level of the AKSZ construction, for both AKSZ topological membrane theories which we derived in §3.2. In each case the reduction leads to a new AKSZ formulation for the topological A-model which differs from the Poisson sigma-model.

4.1 Dimensional reduction of AKSZ membrane sigma-models

We begin by applying a canonical transformation as described in §2.1. Here we will only use infinitesimal canonical transformations, which act on functions \mathbf{f} on the phase space \mathcal{M} as

$$\mathbf{f} \mapsto {}^\alpha \mathbf{f} = \mathbf{f} + \varepsilon (\mathbf{f}, \alpha)_{\text{BV}} , \quad (4.1)$$

where ε is an infinitesimal parameter and α is a fermionic functional of the fields with ghost number -1 . We perform such a canonical transformation on our two AKSZ membrane actions to induce kinetic terms, which will be used for dimensional reduction.

For the first AKSZ membrane action (3.17), the fermionic functional we choose is

$$\alpha = \int_{T[1]\Sigma_3} d^3\hat{z} (\zeta_I \mathbf{D}X^I + \mathbf{N}_I \mathbf{D}\eta^I) , \quad (4.2)$$

where as previously the superworldvolume differential is $\mathbf{D} = \theta^\mu \partial_\mu$. Calculating the BV bracket $(\mathbf{S}_{G_2, \text{I}}, \alpha)_{\text{BV}}$ term by term we get the BRST-equivalent action

$$\begin{aligned} {}^\alpha \mathbf{S}_{G_2, \text{I}} &= \mathbf{S}_{G_2, \text{I}} + \varepsilon (\mathbf{S}_{G_2, \text{I}}, \alpha)_{\text{BV}} \\ &= \int_{T[1]\Sigma_3} d^3\hat{z} \left(\mathbf{F}_I \xi^I + \mathbf{B}^I \mathbf{G}_I + \varepsilon (\mathbf{F}_I \mathbf{D}X^I + \xi^I \mathbf{D}\zeta_I + \mathbf{B}^I \mathbf{D}\mathbf{N}_I - \mathbf{G}_I \mathbf{D}\eta^I) \right) . \end{aligned} \quad (4.3)$$

Similar considerations apply to the second action (3.23): If we restrict the functionals and hence also the action to half of the fields \mathbf{F} , \mathbf{X} , $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, we get the fermionic functional of the canonical transformation $\boldsymbol{\alpha} = \int_{T[1]\Sigma_3} d^3\hat{z} \, \boldsymbol{\zeta}_I \mathbf{D}\mathbf{X}^I$ which gives us the BRST-equivalent action

$$\alpha S_{G_2, \text{II}} = \int_{T[1]\Sigma_3} d^3\hat{z} \left(\mathbf{F}_I \boldsymbol{\xi}^I + \varepsilon (\mathbf{F}_I \mathbf{D}\mathbf{X}^I + \boldsymbol{\xi}^I \mathbf{D}\boldsymbol{\zeta}_I) \right). \quad (4.4)$$

Now let us turn to the dimensional reduction of the AKSZ membrane sigma-models. We assume that the target and worldvolume manifolds are products $M_7 = M_6 \times S^1$ and $\Sigma_3 = \Sigma_2 \times S^1$, where the coordinates of the target and worldvolume circles are indexed by $I = 7$ and $\mu = t$ respectively. We use again the expansion (2.81) of an arbitrary superfield $\phi \in \mathcal{M}$. In terms of expanded superfields, the symplectic structure is given by

$$\begin{aligned} \omega_{3, \text{I}} = \int_{T[1]\Sigma_2} d^2\hat{z} \int_{S^1} dt \left(-\delta\tilde{\mathbf{F}}_I \delta\mathbf{X}_t^I - \delta(\mathbf{F}_t)_I \delta\tilde{\mathbf{X}}^I - \delta\tilde{\boldsymbol{\zeta}}_I \delta\boldsymbol{\xi}_t^I + \delta(\boldsymbol{\zeta}_t)_I \delta\tilde{\boldsymbol{\xi}}^I \right. \\ \left. - \delta\tilde{\mathbf{G}}_I \delta\boldsymbol{\eta}_t^I + \delta(\mathbf{G}_t)_I \delta\tilde{\boldsymbol{\eta}}^I - \delta\tilde{\mathbf{N}}_I \delta\mathbf{B}_t^I - \delta(\mathbf{N}_t)_I \delta\tilde{\mathbf{B}}^I \right), \end{aligned} \quad (4.5)$$

and the action (4.3) by

$$\begin{aligned} \alpha S_{G_2, \text{I}} = \int_{T[1]\Sigma_2} d^2\hat{z} \int_{S^1} dt \left(\tilde{\mathbf{F}}_I \boldsymbol{\xi}_t^I - (\mathbf{F}_t)_I \tilde{\boldsymbol{\xi}}^I + \tilde{\mathbf{B}}^I (\mathbf{G}_t)_I - \mathbf{B}_t^I \tilde{\mathbf{G}}_I \right. \\ + \varepsilon \left(\tilde{\mathbf{F}}_I \tilde{\mathbf{D}}\mathbf{X}_t^I + \tilde{\mathbf{F}}_I \partial_t \tilde{\mathbf{X}}^I - (\mathbf{F}_t)_I \tilde{\mathbf{D}}\tilde{\mathbf{X}}^I + \tilde{\boldsymbol{\xi}}_I \tilde{\mathbf{D}}\boldsymbol{\zeta}_t^I - \tilde{\boldsymbol{\xi}}_I \partial_t \tilde{\boldsymbol{\zeta}}^I + (\boldsymbol{\zeta}_t)_I \tilde{\mathbf{D}}\tilde{\boldsymbol{\zeta}}^I \right. \\ \left. - \tilde{\mathbf{G}}_I \tilde{\mathbf{D}}\boldsymbol{\eta}_t^I + \tilde{\mathbf{G}}_I \partial_t \tilde{\boldsymbol{\eta}}^I - (\mathbf{G}_t)_I \tilde{\mathbf{D}}\tilde{\boldsymbol{\eta}}^I + \tilde{\mathbf{B}}_I \tilde{\mathbf{D}}\mathbf{N}_t^I + \tilde{\mathbf{B}}_I \partial_t \tilde{\mathbf{N}}^I - (\mathbf{B}_t)_I \tilde{\mathbf{D}}\tilde{\mathbf{N}}^I \right). \end{aligned} \quad (4.6)$$

We choose $\tilde{\mathbf{F}}$, \mathbf{X}_t , $\tilde{\boldsymbol{\phi}}$, $\boldsymbol{\xi}_t$, $\tilde{\mathbf{G}}$, $\boldsymbol{\eta}_t$, $\tilde{\mathbf{N}}$ and \mathbf{B}_t to be the ultraviolet fields, and the rest to be the infrared fields. We define the gauge fixing condition as the Lagrangian submanifold \mathcal{L} defined by $\mathbf{X}_t = \boldsymbol{\eta}_t = \boldsymbol{\xi}_t = \mathbf{B}_t = 0$, and then integrate out the remaining ultraviolet fields. This leads to the conditions $\partial_t \tilde{\mathbf{X}} = \partial_t \tilde{\boldsymbol{\xi}} = \partial_t \tilde{\boldsymbol{\eta}} = \partial_t \tilde{\mathbf{B}} = 0$, so these fields do not depend on t . We also integrate out all of the fields with $I = 7$ index, and introduce new fields

$$\chi_i = - \int_{S^1} dt (\mathbf{F}_t)_i, \quad \mathbf{p}_i = \int_{S^1} dt (\boldsymbol{\zeta}_t)_i, \quad \mathbf{h}_i = \int_{S^1} dt (\mathbf{G}_t)_i \quad \text{and} \quad \mathbf{n}_i = - \int_{S^1} dt (\mathbf{N}_t)_i, \quad (4.7)$$

and

$$\mathbf{X}^i = \tilde{\mathbf{X}}^i, \quad \mathbf{q}^i = \tilde{\boldsymbol{\xi}}^i, \quad \boldsymbol{\eta}^i = \tilde{\boldsymbol{\eta}}^i \quad \text{and} \quad \mathbf{b}^i = \tilde{\mathbf{B}}^i, \quad (4.8)$$

where we used the index notation $I = (i, 7)$ with $i = 1, \dots, 6$ the coordinate directions along M_6 . Our effective action is then

$$S_{G_2, \text{I}}^{\text{eff}} = \int_{T[1]\Sigma_2} d^2\hat{z} \left(\chi_i \mathbf{q}^i + \mathbf{b}^i \mathbf{h}_i + \varepsilon (\chi_i \mathbf{D}\mathbf{X}^i + \mathbf{q}^i \mathbf{D}\mathbf{p}_i - \mathbf{h}_i \mathbf{D}\boldsymbol{\eta}^i - \mathbf{b}^i \mathbf{D}\mathbf{n}_i) \right), \quad (4.9)$$

and the new symplectic form is

$$\omega_{2, \text{I}} = \int_{T[1]\Sigma_2} d^2\hat{z} \left(\delta\chi_i \delta\mathbf{X}^i + \delta\mathbf{p}_i \delta\mathbf{q}^i + \delta\mathbf{h}_i \delta\boldsymbol{\eta}^i + \delta\mathbf{n}_i \delta\mathbf{b}^i \right). \quad (4.10)$$

We now perform another infinitesimal canonical transformation with the same parameter ε and the fermion

$$\boldsymbol{\alpha}' = - \int_{T[1]\Sigma_2} d^2\hat{z} \, \mathbf{p}_i \mathbf{D}\mathbf{X}^i - \int_{\Sigma_2} d^2\sigma \left(n_i^{(0)} d\eta_i^{(1)} - n_i^{(1)} d\eta_i^{(0)} \right) \quad (4.11)$$

in order to eliminate the kinetic terms. In this way we arrive at the action

$$\mathcal{S}_{\text{A,I}} = \int_{T[1]\Sigma_2} d^2\hat{z} (\chi_i \mathbf{q}^i + \mathbf{b}^i h_i) . \quad (4.12)$$

If we restrict this construction and the dimensional reduction to half of the fields \mathbf{F} , \mathbf{X} , $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, we arrive at the action for the dimensional reduction of our second AKSZ membrane model in the form

$$\mathcal{S}_{\text{A,II}} = \int_{T[1]\Sigma_2} d^2\hat{z} \chi_i \mathbf{q}^i . \quad (4.13)$$

In the following we will introduce AKSZ constructions which give the actions (4.12) and (4.13), and then relate them to the topological A-model via suitable choices of Lagrangian submanifolds $\mathcal{L} \subset \mathcal{M}$ (or gauge fixing). For this, we equip $M_7 = M_6 \times S^1$ with a direct product metric, where M_6 is a six-dimensional Riemannian manifold with $SU(3)$ -structure, and write the G_2 -structure on M_7 as

$$\Phi = dX^7 \wedge B + \rho , \quad (4.14)$$

where B is an almost Kähler form of type $(1,1)$ with respect to the almost complex structure defined by the three-form ρ on M_6 . If B and ρ are independent of X^7 , then $d\Phi = 0$ implies $dB = d\rho = 0$ and M_6 is a Calabi-Yau threefold, as in the A-model topological string theory, where ρ is the real part of the global holomorphic three-form Ω on M_6 . However, for the purposes of our ensuing AKSZ constructions only the Kähler class of the Calabi-Yau structure is required, as in [7]. In particular, double dimensional reduction on a circle of the flux deformation $I_{\Sigma_3, \Phi}$ along the lines of §2.7 gives the B -field coupling $I_{\Sigma_2, wB}$ for the topological string, whose AKSZ construction is given by the Poisson sigma-model of §2.3. Hence in what follows we shall only require that M_6 be a Kähler manifold.

4.2 The topological A-model

Let us briefly review the topological A-model, whose Mathai-Quillen formalism is given in e.g. [4]. It is defined by maps $X^i = (X^a, X^{\bar{a}})$ from the worldsheet Σ_2 to the Kähler manifold M_6 , where $a = 1, 2, 3$ are complex target space indices and we use local complex coordinates $\sigma = (z, \bar{z})$ on the Riemann surface Σ_2 . We further introduce ghost fields $(\chi_{\bar{z}}, \chi_z^{\bar{a}}, \psi^a, \psi^{\bar{a}})$ with ghost number $(-1, -1, 1, 1)$. The action of the topological A-model is then

$$I_{\text{A}} = \int_{\Sigma_2} d^2z \left(g_{a\bar{b}} \partial_{\bar{z}} X^a \partial_z X^{\bar{b}} + i g_{a\bar{b}} (\chi_{\bar{z}}^a \nabla_z \psi^{\bar{b}} + \chi_z^{\bar{b}} \nabla_{\bar{z}} \psi^a) - R_{a\bar{b}c\bar{d}} \chi_{\bar{z}}^a \chi_z^{\bar{b}} \psi^c \psi^{\bar{d}} \right) , \quad (4.15)$$

where $g_{a\bar{b}}$ is the Kähler metric which obeys the Kähler identity $\partial_a g_{b\bar{c}} = \partial_{\bar{b}} g_{a\bar{c}}$ and its complex conjugate $\partial_{\bar{a}} g_{b\bar{c}} = \partial_{\bar{c}} g_{b\bar{a}}$. The Levi-Civita connection is $\nabla_z \psi^{\bar{a}} = \partial_z \psi^{\bar{a}} + \Gamma_{\bar{b}\bar{c}}^{\bar{a}} \psi^{\bar{b}} \partial_z X^{\bar{c}}$ and its complex conjugate $\nabla_{\bar{z}} \psi^a = \partial_{\bar{z}} \psi^a + \Gamma_{bc}^a \psi^b \partial_{\bar{z}} X^c$. The complex Christoffel symbols are $\Gamma_{bc}^a = g^{a\bar{d}} \Gamma_{\bar{d}bc}$ and $\Gamma_{\bar{b}\bar{c}}^{\bar{a}} = g^{\bar{a}d} \Gamma_{d\bar{b}\bar{c}}$, where $\Gamma_{a\bar{b}\bar{c}} = \partial_{\bar{b}} g_{a\bar{c}}$ and $\Gamma_{\bar{a}bc} = \partial_b g_{c\bar{a}}$. The Riemann tensor is $R_{a\bar{b}c\bar{d}} = -g_{a\bar{e}} \partial_c \Gamma_{\bar{b}\bar{d}}^{\bar{e}}$. The action (4.15) is invariant under the BRST transformations

$$\delta X^a = i \psi^a , \quad \delta X^{\bar{a}} = i \psi^{\bar{a}} , \quad \delta \psi^a = 0 \quad \text{and} \quad \delta \psi^{\bar{a}} = 0 , \quad (4.16)$$

together with

$$\delta \chi_{\bar{z}}^a = -\partial_{\bar{z}} X^a - i \Gamma_{bc}^a \psi^b \chi_{\bar{z}}^c \quad \text{and} \quad \delta \chi_z^{\bar{a}} = -\partial_z X^{\bar{a}} - i \Gamma_{\bar{b}\bar{c}}^{\bar{a}} \psi^{\bar{b}} \chi_z^{\bar{c}} . \quad (4.17)$$

The fixed point locus of the BRST charge is the space of holomorphic maps $X : \Sigma_2 \rightarrow M_6$, which are worldsheet instantons.

Let us now reformulate the topological A-model with a linearizing auxiliary field, analogously to what we did in §3.1 for the Mathai-Quillen membrane sigma-model. We introduce two fields b_z^a and $b_z^{\bar{a}}$ with ghost number 0, and the new BRST transformations given by (4.16) together with

$$\delta\chi_z^a = b_z^a, \quad \delta\chi_z^{\bar{a}} = b_z^{\bar{a}}, \quad \delta b_z^a = 0 \quad \text{and} \quad \delta b_z^{\bar{a}} = 0. \quad (4.18)$$

The action

$$S_A = \delta\Psi_A, \quad (4.19)$$

with the gauge fixing fermion

$$\begin{aligned} \Psi_A = - \int_{\Sigma_2} d^2z \left(g_{a\bar{b}} (\chi_z^a \partial_z X^{\bar{b}} + \chi_z^{\bar{a}} \partial_z X^b) + \frac{1}{2} g_{a\bar{b}} (\chi_z^a b_z^{\bar{b}} + \chi_z^{\bar{a}} b_z^b) \right. \\ \left. + \frac{i}{2} \Gamma_{a\bar{b}\bar{c}} \psi^{\bar{b}} \chi_z^{\bar{c}} \chi_z^a + \frac{i}{2} \Gamma_{abc} \psi^b \chi_z^c \chi_z^{\bar{a}} \right), \end{aligned} \quad (4.20)$$

reduces to the action (4.15) after using the equations of motion of the auxiliary fields b_z^a and $b_z^{\bar{a}}$ which give

$$b_z^a = -\partial_z X^a - i\Gamma_{bc}^a \psi^b \chi_z^c \quad \text{and} \quad b_z^{\bar{a}} = -\partial_z X^{\bar{a}} - i\Gamma_{\bar{b}\bar{c}}^{\bar{a}} \psi^{\bar{b}} \chi_z^{\bar{c}}. \quad (4.21)$$

4.3 BV formulation and AKSZ constructions

If we define the antifields as

$$X_a^+ = \frac{\delta\Psi_A}{\delta X^a}, \quad X_{\bar{a}}^+ = \frac{\delta\Psi_A}{\delta X^{\bar{a}}}, \quad (\chi_a^+)_z = \frac{\vec{\delta}\Psi_A}{\delta\chi_z^a} \quad \text{and} \quad (\chi_{\bar{a}}^+)_{\bar{z}} = \frac{\vec{\delta}\Psi_A}{\delta\chi_z^{\bar{a}}}, \quad (4.22)$$

we can rewrite (4.19) as a BV-type action

$$S_A = \int_{\Sigma_2} d^2z \left(\psi^a X_a^+ + \psi^{\bar{a}} X_{\bar{a}}^+ + b_z^a (\chi_a^+)_z + b_z^{\bar{a}} (\chi_{\bar{a}}^+)_{\bar{z}} \right). \quad (4.23)$$

In the following we give two new AKSZ constructions for the topological A-model, which each differ from the Poisson sigma-model.

AKSZ construction I. Our first AKSZ construction for the topological A-model is analogous to the first AKSZ membrane sigma-model in §3.2. The source dg-manifold is the superworldsheet $\mathcal{W} = T[1]\Sigma_2$, while the target symplectic dg-manifold is $\mathcal{M} = T^*[1]T[-1]T[1]M_6$, where M_6 is a Kähler manifold. The base coordinates in $T[-1]T[1]M_6$ are (X^i, q^i, b^i, η^i) with degree $(0, 1, 0, -1)$, where X^i are associated to the coordinates of M_6 . The graded fiber coordinates are (χ_i, p_i, n_i, h_i) with degree $(1, 0, 1, 2)$. The canonical symplectic structure of degree 1 on the target superspace \mathcal{M} is

$$\omega_{2,I} = d\chi_i \wedge dX^i + dp_i \wedge dq^i + dh_i \wedge d\eta^i + dn_i \wedge db^i. \quad (4.24)$$

This gives the same BV symplectic structure on the space \mathcal{M} of superfields as in (4.10), and the AKSZ action is (4.12).

We choose the Lagrangian submanifold $\mathcal{L} \subset \mathcal{M}$ given by

$$\chi^{(0)} = \chi^{(1)} = 0, \quad p^{(0)} = p^{(1)} = 0, \quad n^{(0)} = n^{(1)} = 0 \quad \text{and} \quad h^{(0)} = h^{(1)} = 0. \quad (4.25)$$

Writing the coordinate indices of the Kähler manifold M_6 as before in complex notation $i = (a, \bar{a})$, where $a = 1, 2, 3$, and the complex coordinates on the worldsheet Σ_2 as (z, \bar{z}) , we define the component fields

$$\begin{aligned} X^{(0)i} &= X^i, & \chi_i^{(2)} &= X_i^+, & q^{(0)i} &= \psi^i & \text{and} & & p_i^{(2)} &= -\psi_i^+, \\ b^{(0)a} &= b_{\bar{z}}^a, & b^{(0)\bar{a}} &= b_{\bar{z}}^{\bar{a}}, & n_a^{(2)} &= (b_a^+)_z & \text{and} & & n_{\bar{a}}^{(2)} &= (b_{\bar{a}}^+)_{\bar{z}}, \\ \eta^{(0)a} &= \chi_{\bar{z}}^a, & \chi^{(0)\bar{a}} &= \chi_{\bar{z}}^{\bar{a}}, & h_a^{(2)} &= -(\chi_a^+)_z & \text{and} & & h_{\bar{a}}^{(2)} &= -(\chi_{\bar{a}}^+)_{\bar{z}}. \end{aligned} \quad (4.26)$$

With this notation, the restriction of the AKSZ action (4.12) to \mathcal{L} yields the BV action (4.23) as $\mathbf{S}_{A,I}|_{\mathcal{L}} = -S_A$, and it can be gauge fixed further to the A-model action with gauge fixing fermion $-\Psi_A$ from (4.20).

AKSZ construction II. We introduce a second AKSZ construction for the topological A-model, which is the analogue of the second AKSZ membrane sigma-model in §3.2. We start with the same source dg-manifold $\mathcal{W} = T[1]\Sigma_2$ as in the previous construction, but now we choose $\mathcal{M} = T^*[1]T[1]M_6$ to be the target QP-manifold of degree 1 with coordinates (X^i, χ_i, q^i, p_i) with degree $(0, 1, 1, 0)$. The symplectic structure

$$\omega_{2,\Pi} = d\chi_i \wedge dX^i + dp_i \wedge dq^i \quad (4.27)$$

is the restriction of (4.24). The AKSZ action is also the restriction (4.13).

We introduce the component fields

$$\begin{aligned} X^{(0)i} &= X^i, & (\chi_i^{(2)})_{z\bar{z}} &= X_i^+, & q^{(0)i} &= \psi^i & \text{and} & & (q_i^{(2)})_{z\bar{z}} &= \psi_i^+, \\ X_{\bar{z}}^{(1)a} &= \chi_{\bar{z}}^a, & (\chi_a^{(1)})_z &= -(\chi_a^+)_z, & X_z^{(1)\bar{a}} &= \chi_z^{\bar{a}} & \text{and} & & (\chi_{\bar{a}}^{(1)})_{\bar{z}} &= (\chi_{\bar{a}}^+)_{\bar{z}}, \\ q_{\bar{z}}^{(1)a} &= b_{\bar{z}}^a, & (p_a^{(1)})_z &= -(b_a^+)_z, & q_z^{(1)\bar{a}} &= b_z^{\bar{a}} & \text{and} & & (p_{\bar{a}}^{(1)})_{\bar{z}} &= (b_{\bar{a}}^+)_{\bar{z}}, \end{aligned} \quad (4.28)$$

and choose the Lagrangian submanifold defined by

$$\chi^{(0)i} = 0, \quad p_i^{(0)} = 0, \quad (\chi_a^{(1)})_{\bar{z}} = (\chi_{\bar{a}}^{(1)})_z = 0 \quad \text{and} \quad (p_a^{(1)})_{\bar{z}} = (p_{\bar{a}}^{(1)})_z = 0. \quad (4.29)$$

This yields the same BV action $-S_A$ from (4.23), which gives the A-model action with gauge fixing fermion $-\Psi_A$ from (4.20). Note that in neither of these AKSZ constructions does the target dg-manifold coincide with that of the Poisson sigma-model from §2.3 associated to string fields $X : \Sigma_2 \rightarrow M_6$.

4.4 Dimensional reduction from the standard Courant sigma-model

The first AKSZ construction of the A-model in §4.3 can be embedded into the standard Courant sigma-model, which is a membrane theory, in a similar way as we embedded our AKSZ membrane sigma-models into the standard 2-Courant sigma-model, which is a threebrane theory, in §3.4. For this, let us consider the standard Courant sigma-model from §2.4 on a product worldvolume $\Sigma_3 = \Sigma_2 \times S^1$, and assume that our superfields do not depend on the extra coordinate of S^1 . The AKSZ action of the standard standard Courant sigma-model is given by (2.55) and the BV symplectic form by (2.54). After integration over the extra supercoordinates on $T[1]S^1$ and a relabelling of superfields, we arrive at the AKSZ action

$$\mathbf{S}_{2,\text{red}} = \int_{T[1]\Sigma_2} d^2\hat{z} \left(\chi_i q^i + b^i h_i - \chi_i D X^i - h_i D \eta^i - n_i D b^i - p_i D q^i \right), \quad (4.30)$$

and the BV symplectic form

$$\omega_{2,\text{red}} = \int_{T[1]\Sigma_2} d^2\hat{z} \left(\delta\chi_i \delta X^i + \delta p_i \delta q^i + \delta h_i \delta \eta^i + \delta n_i \delta b^i \right). \quad (4.31)$$

This symplectic form is the same as that of the A-model in (4.10), and the AKSZ action reduces to the A-model action (4.12) if we set the kinetic terms to zero by definition or via gauge fixing.

5 AKSZ theory for supersymmetric quantum mechanics

In this section we continue the dimensional reduction procedure one final time, and reduce our second AKSZ construction of the A-model to an AKSZ formulation for supersymmetric quantum mechanics. We have seen in §3.2 and §4.3 that both the topological A-model and the topological membrane sigma-models on G_2 -manifolds have similar AKSZ constructions. Following the same procedure as before we give an analogous AKSZ construction for supersymmetric quantum mechanics.

5.1 Dimensional reduction of the A-model

We start with the canonically transformed action from (4.9) restricted to the fields of the second AKSZ construction:

$$S_{G_2,\text{II}}^{\text{eff}} = \int_{T[1]\Sigma_2} d^2\hat{z} \left(\chi_i q^i + \varepsilon (\chi_i D X^i + q^i D p_i) \right), \quad (5.1)$$

and the corresponding symplectic structure from (4.10):

$$\omega_{2,\text{II}} = \int_{T[1]\Sigma_2} d^2\hat{z} \left(\delta\chi_i \delta X^i + \delta p_i \delta q^i \right). \quad (5.2)$$

We apply Losev's trick from §2.6 and use the same notation as in (2.81) to calculate the reduction on a product source space $\Sigma_2 = S^1 \times \Sigma_1$, where we distinguish the circle $\Sigma_1 = S^1$ along which the dimensional reduction takes place. We choose $\widetilde{\chi}_i, \widetilde{q}^i, X_t^i$ and $(p_t)_i$ to be the ultraviolet fields, and we set the gauge $X_t^i = 0$ and $\widetilde{q}^i = 0$. After integrating out the ultraviolet fields, we obtain the effective action

$$S_{A,\text{II}}^{\text{eff}} = \int_{T[1]S^1} d\hat{z} \left(B_i \xi^i + \varepsilon (-B_i D X^i - \xi^i D \eta_i) \right), \quad (5.3)$$

and the symplectic structure

$$\omega_1 = \int_{T[1]S^1} d\hat{z} \left(\delta B_i \delta X^i + \delta \eta_i \delta \xi^i \right), \quad (5.4)$$

where we relabeled the fields as

$$B_i = - \int_{\Sigma_1} dt (\chi_t)_i, \quad X_i = \widetilde{X}^i, \quad \eta_i = - \int_{\Sigma_1} dt (p_t)_i \quad \text{and} \quad \xi^i = -\widetilde{q}^i, \quad (5.5)$$

and these new fields are independent of the coordinate t of Σ_1 .

The infinitesimal canonical transformation (4.1) with the fermionic functional

$$\alpha = \int_{T[1]S^1} d\hat{z} \eta_i D X^i \quad (5.6)$$

gives the action

$$S_{\text{SQM}} = \int_{T[1]S^1} d\hat{z} \, B_i \xi^i. \quad (5.7)$$

We will see in §5.3 below that this action gives an AKSZ formulation of supersymmetric quantum mechanics. Nothing we discuss in this section depends on the target space Kähler structure nor even on its dimensionality, and the reduction of the topological sigma-model described here applies to generic maps whose target is any Riemannian manifold.

5.2 Supersymmetric quantum mechanics

Supersymmetric quantum mechanics provides a simple example of a topological field theory; its Mathai-Quillen formalism can be found in e.g. [3–5]. The target space is a Riemannian manifold M with metric g and the parameter manifold is just a compact worldline S^1 . The local coordinates of the mapping space $LM := \text{Map}(S^1, M)$ are $\{x^i(\tau)\}$ with $\tau \in [0, 1]$ and $x^i(0) = x^i(1)$, and so they parameterize (smooth) loops in M . We furthermore define two fermionic fields $\psi^i(\tau)$ and $\bar{\psi}_i(\tau)$ with ghost number 1 and -1 , respectively. The action of supersymmetric quantum mechanics is

$$I_{\text{SQM}} = \int_{S^1} d\tau \left(\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + i \bar{\psi}_i \nabla_\tau \psi^i - \frac{1}{4} R^{ij}_{kl} \bar{\psi}_i \bar{\psi}_j \psi^k \psi^l \right), \quad (5.8)$$

where a dot denotes a τ -derivative, $\nabla_\tau \psi^i = \dot{\psi}^i + \Gamma^i_{jk} \psi^j \dot{x}^k$ is defined by the action of the Levi-Civita connection ∇ of the metric g pulled back to the loop via the map x , and R is the associated Riemann tensor. The action (5.8) is invariant under the BRST transformations

$$\delta x^i = \psi^i, \quad \delta \psi^i = 0 \quad \text{and} \quad \delta \bar{\psi}_i = i g_{ij} \dot{x}^j + \Gamma^k_{ij} \psi^j \bar{\psi}_k, \quad (5.9)$$

which is only nilpotent on-shell, and it is BRST-exact on-shell:

$$I_{\text{SQM}} = \delta \Psi'_{\text{SQM}} \quad \text{with} \quad \Psi'_{\text{SQM}} = -\frac{i}{2} \int_{S^1} d\tau \, \bar{\psi}_i \dot{x}^i. \quad (5.10)$$

The set of δ -fixed points is the space of instantons, i.e. the constant loops $x^i(\tau)$, which can be identified with the target space M .

We follow the same procedure as in §3.1 and §4.2 to reformulate supersymmetric quantum mechanics using a linearizing auxiliary field b_i with ghost number 0. The BRST transformations with the field b_i are given by

$$\delta x^i = \psi^i, \quad \delta \psi^i = 0, \quad \delta \bar{\psi}_i = b_i \quad \text{and} \quad \delta b_i = 0, \quad (5.11)$$

and they are nilpotent off-shell. The action

$$\delta \Psi'_{\text{SQM}} = -\frac{i}{2} \int_{S^1} d\tau \, (b_i \dot{x}^i - \bar{\psi}_i \dot{\psi}^i) \quad (5.12)$$

is invariant under these new BRST transformations, and it reduces to the action (5.8) if we impose the constraint

$$b_i = i g_{ij} \dot{x}^j - \Gamma^k_{ij} \bar{\psi}_k \psi^j \quad (5.13)$$

as gauge fixing. In the language of the BRST formulation, this means that we choose the gauge fixing fermion as

$$\Psi_{\text{SQM}} = - \int_{S^1} d\tau \, \bar{\psi}_i \left(i \dot{x}^i + \frac{1}{2} g^{jl} \Gamma^i_{lk} \bar{\psi}_j \psi^k - \frac{1}{2} g^{ij} b_j \right). \quad (5.14)$$

The BRST variation of (5.14) gives us the action

$$S_{\text{SQM}} = \delta \Psi_{\text{SQM}} = \int_{S^1} d\tau \left(i \bar{\psi}_i \dot{\psi}^i - i b_i \dot{x}^i + g^{il} \Gamma_{kl}^j \bar{\psi}_j \psi^k b_i + \frac{1}{2} g^{ij} b_i b_j - \frac{1}{2} \partial_k (g^{jm} \Gamma_{ml}^i) \bar{\psi}_i \bar{\psi}_j \psi^k \psi^l \right). \quad (5.15)$$

The equation of motion for b_i gives the same field redefinition as in (5.13), and using this we find that the action (5.15) is classically equivalent to the action (5.8).

5.3 AKSZ construction

Following the procedure in §3.2 and §4.3, we give an AKSZ formulation of supersymmetric quantum mechanics which reduces to the action (5.8) after gauge fixing and eliminating the auxiliary field b_i . Our source dg-manifold is $\mathcal{W} = T[1]S^1$ and the target symplectic dg-manifold is $\mathcal{M} = T^*(T[1]M)$, where M is a Riemannian manifold with metric g . Denote the degree 0 and 1 coordinates of $T[1]M$ by X^i and ξ^i , respectively, and their cotangent coordinates by B_i and η_i with degree 0 and -1 , respectively. The canonical symplectic structure on $\mathcal{M} = T^*(T[1]M)$ is

$$\omega_1 = dB_i \wedge dX^i + d\eta_i \wedge d\xi^i, \quad (5.16)$$

which gives the same symplectic form on the mapping space of superfields \mathcal{M} as in (5.4). The AKSZ superfields are expanded as

$$\begin{aligned} X^i &= x^i - b^{+i} \theta, \\ B_i &= -b_i + x_i^+ \theta, \\ \xi^i &= -\psi^i + \bar{\psi}^{+i} \theta, \\ \eta_i &= \bar{\psi}_i - \psi_i^+ \theta, \end{aligned} \quad (5.17)$$

where the superworldline coordinate θ has degree 1. Our choice for the AKSZ action is the same as that in (5.7) which was obtained from the dimensional reduction of the A-model:

$$\mathcal{S}_{\text{SQM}} = \int_{T[1]S^1} d\hat{z} B_i \xi^i = - \int_{S^1} d\tau (\psi^i x_i^+ + b_i \bar{\psi}^{+i}), \quad (5.18)$$

and it trivially solves the classical master equation $(\mathcal{S}_{\text{SQM}}, \mathcal{S}_{\text{SQM}})_{\text{BV}} = 0$. The BV–BRST transformations¹² are generated by the cohomological vector field given by the BV bracket $\mathcal{Q}_{\text{SQM}} = (\mathcal{S}_{\text{SQM}}, \cdot)_{\text{BV}}$ and read as

$$\begin{aligned} \mathcal{Q}_{\text{SQM}} x^i &= \psi^i & \text{and} & & \mathcal{Q}_{\text{SQM}} \psi_i^+ &= x_i^+, \\ \mathcal{Q}_{\text{SQM}} \psi^i &= 0 & \text{and} & & \mathcal{Q}_{\text{SQM}} x_i^+ &= 0, \\ \mathcal{Q}_{\text{SQM}} \bar{\psi}_i &= b_i & \text{and} & & \mathcal{Q}_{\text{SQM}} b^{+i} &= -\bar{\psi}^{+i}, \\ \mathcal{Q}_{\text{SQM}} b_i &= 0 & \text{and} & & \mathcal{Q}_{\text{SQM}} \bar{\psi}^{+i} &= 0. \end{aligned} \quad (5.19)$$

The nilpotent fermionic symmetry \mathcal{Q}_{SQM} acts trivially on the AKSZ action \mathcal{S}_{SQM} .

¹²The action \mathcal{S}_{SQM} is also invariant under the transformations $\delta x_i^+ = b_i$, $\delta \bar{\psi}^{+i} = \psi^i$, $\delta \psi^i = 0$ and $\delta b_i = 0$, and under the transformations $\delta x_i^+ = 0$, $\delta \bar{\psi}^{+i} = 0$, $\delta \psi^i = \bar{\psi}^{+i}$ and $\delta b_i = -x_i^+$, but our transformations do not include these.

We reduce the action \mathcal{S}_{SQM} to I_{SQM} after gauge fixing. We choose the same gauge fixing fermion $-\Psi_{\text{SQM}}$ as in (5.14). The pertinent antifields are given by

$$\begin{aligned} x_i^+ &= -\frac{\delta\Psi_{\text{SQM}}}{\delta x^i} = -i\dot{\psi}_i + \frac{1}{2}\partial_i(\Gamma_{ml}^j g^{mk})\bar{\psi}_j\bar{\psi}_k\psi^l - \frac{1}{2}\partial_i g^{jk}\bar{\psi}_j b_k, \\ \bar{\psi}^{+i} &= -\frac{\overrightarrow{\delta\Psi_{\text{SQM}}}}{\delta\bar{\psi}_i} = i\dot{x}^i + \Gamma_{lk}^{[i} g^{j]l}\bar{\psi}_j\psi^k - \frac{1}{2}g^{ij}b_j, \end{aligned} \tag{5.20}$$

where the other gauge fixing equations are not important here. Calculating the gauge fixed action of \mathcal{S}_{SQM} we get the action (5.15), which is classically equivalent to I_{SQM} .

6 Conclusions and outlook

In this paper we have constructed BV quantized topological membrane theories on G_2 -manifolds using the AKSZ formulation, which unify the topological membrane theories of [3] and [6]. We have dimensionally reduced them to the A-model, and one of them has been reduced further to supersymmetric quantum mechanics. We also studied the derived bracket of one of our AKSZ topological membrane theories whose target is a derived symplectic dg-manifold with fields of negative degree, which gave an L_∞ -extension of the standard Courant bracket. It would be interesting to study further the consequences of this more complex derived algebroid structure.

We have further proposed a topological threebrane model given by the AKSZ construction, which reduces to our AKSZ membrane theories upon worldvolume dimensional reduction. Its derived bracket is the standard 2-Courant bracket, which appears in exceptional generalized geometry as the antisymmetrization of the generalized Lie derivative, and it is also the induced bracket of anomaly-free current algebras of topological membranes on G_2 -manifolds [11]. We have found that double dimensional reduction on a circle of our threebrane model with G -flux twisting yields the twisted standard Courant sigma-model, which geometrizes the H -flux in type II string theory.

Our constructions are the starting point for the introduction of exceptional generalized geometry [42, 43, 49] and M-theory fluxes [50, 51] for membranes in M-theory described by the AKSZ formalism. The first step towards this goal is our AKSZ threebrane sigma-model with its derived standard 2-Courant bracket. However, implementing non-geometric M-theory fluxes into this setting seems somewhat perplexing. In the string theory setting, T-duality in AKSZ membrane theory acts as a duality between standard and contravariant Courant sigma-models, and also transforms geometric H -flux and non-geometric R -flux into each other [25]. It is tempting to try lifting this T-duality to a U-duality at the level of AKSZ threebrane theory, which transforms our threebrane into another topological threebrane with non-geometric flux. In the case of the Courant sigma-models, the duality interchanges the degree 1 coordinates ψ^i and χ_i , and it is implemented as a canonical transformation given by a bivector and its T-dual two-form B -field. For the 2-Courant sigma-models, it is natural to expect that there similarly exist canonical transformations which implement the interchange between the degree 2 quantities $\psi^i\psi^j$ and χ_i . In this case a trivector and a three-form would arise, which should be related to the trivector and three-form C -field in exceptional generalized geometry. But unfortunately this does not seem to be the case as there are no symplectomorphisms which interchange $\psi^i\psi^j$ and χ_i . Thus implementing U-duality and non-geometric fluxes seems to be far more complicated than in the string theory case.

We close by discussing some open avenues for future investigation. In [52] a closed string on a G_2 -manifold has been proposed as the dual of a topological G_2 membrane, and its quantization at one-loop order is considered in [53], which may be relevant to the quantization of our membrane construction that is of interest when considering its connection to physical string theory (see also [3]). Likewise an open G_2 string theory is introduced in [54], wherein the worldvolume theory

of associative three-cycles has a membrane formulation given by a gauge fixed Chern-Simons theory coupled to normal deformations of the cycle. A further development would be to give an AKSZ construction for this three-cycle theory, and to compare it with our AKSZ topological membrane theories. It would also be interesting to study the topological membrane of [55] in the context of the AKSZ construction. Finally, in the present paper we also derived AKSZ constructions for the A-model, hence one of the applications of our results is to study the possible dualities between the A-model and the B-model at the level of the AKSZ formalism, and in particular to find a realization of S-duality [56] in AKSZ theory. In this respect it would be interesting to study further the three-brane theory of calibrated four-cycles on eight-dimensional $Spin(7)$ -manifolds that we discussed in Section 3.4, which may be relevant to the study of S-duality as in [47].

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